

A GROWTH-FRAGMENTATION MODEL RELATED TO ORNSTEIN-UHLENBECK TYPE PROCESSES

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Abstract

Growth-fragmentation processes describe the evolution of particles that grow and divide randomly as time proceeds. Unlike previous studies, which have focused mainly on the self-similar case, we introduce a new type of growth-fragmentation which is closely related to Lévy driven Ornstein-Uhlenbeck type processes. Our model can be viewed as a generalization of compensated fragmentation processes introduced by Bertoin (Ann. Prob. 2015), or the stochastic counterpart of a family of growth-fragmentation equations. We establish a convergence criterion for a sequence of such growth-fragmentations. We also prove that, under certain conditions, the average size of the particles converges to a stationary distribution as time tends to infinity.

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1 Introduction

Fragmentation processes describe particles that split randomly as time passes, independently one of the others; see [8] for a comprehensive overview. Recently, Bertoin [9, 10] extended fragmentations to *growth-fragmentation processes*, in which a particle may also grow and decay continuously. In both (pure) fragmentations and growth-fragmentations, research has focused on the *self-similar* case, which means the particle system behaves the same when viewed at certain different scales on space and time.

In the present work, we propose a new type of growth-fragmentation that possesses a different scaling property. We name it an *Ornstein-Uhlenbeck (OU) type growth-fragmentation process*, as in such a particle system, informally speaking, each particle splits and grows independently, and its size evolves according to the exponential of an *OU type process* $(Z(t), t \geq 0)$ driven by a Lévy process ξ :

$$Z(t) := e^{-\theta t} Z(0) + \int_0^t e^{-\theta(t-s)} d\xi(s), \quad t \geq 0, \quad (1.1)$$

where $\theta \in \mathbb{R}$ and the integral is defined in the sense of a stochastic integral, as the Lévy process ξ is a semimartingale. If ξ is a Brownian motion, then Z is a well-known Gaussian OU process. Our model is partially motivated by a recent work [5] (see also a related work [30]), results in which imply that a certain OU type growth-fragmentation naturally arises in dynamical percolation on an infinite recursive tree; see Section 5 for details. Besides this motivation, our model may have potential applications, as OU type processes are

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widely applied in various domains: in biology, they are used in a neuronal model with signal-dependent noise [27]; in finance, they are used in an option price model with stochastic volatility [3, 4], to name just a few.

We now give a more precise description of OU type growth-fragmentations. Let c_o^\downarrow be the space of decreasing null sequences (that converge to 0), endowed with the ℓ^∞ -norm. An OU type growth-fragmentation process is a c_o^\downarrow -valued càdlàg Markov process

$$\mathbf{X} = \left(\mathbf{X}(t) := (X_1(t), X_2(t), \dots), \quad t \geq 0 \right),$$

where $\mathbf{X}(t)$ is viewed as the decreasing sequence of the size of the particles alive at time t . For every $x \in \mathbb{R}_+ = (0, \infty)$, let \mathbf{P}_x denote the law of \mathbf{X} with initial value $\mathbf{X}(0) = (x, 0, \dots) \in c_o^\downarrow$. The process \mathbf{X} further satisfies the following properties:

- (P1) (The branching property) For every sequence $\mathbf{x} = (x_1, x_2, \dots) \in c_o^\downarrow$, the process \mathbf{X} starting from $\mathbf{X}(0) = \mathbf{x}$ has the same law as the union of the elements, arranged in decreasing order, of a family of independent OU type growth-fragmentations $(\mathbf{X}^{[i]})_{i \geq 1}$, where each $\mathbf{X}^{[i]}$ has distribution \mathbf{P}_{x_i} .
- (P2) (The OU property) There exists an certain index $\theta \in \mathbb{R}$, such that for every $x \in \mathbb{R}_+$, the distribution of the rescaled process $(x^{\exp(-\theta t)} \mathbf{X}(t))_{t \geq 0}$ under \mathbf{P}_1 is \mathbf{P}_x .

The branching property indicates that the fragments evolve independently one of the others. The OU property is due to the scaling property of the exponential of an OU type process (a direct consequence of (1.1)). For comparison, recall that a *self-similar growth-fragmentation* \mathbf{Y} (in particular it can be a self-similar fragmentation) fulfills the same branching property, but a different scaling property: namely, for a certain index $\alpha \in \mathbb{R}$, the rescaled process $(x \mathbf{Y}(x^\alpha t))_{t \geq 0}$ under \mathbf{P}_1 is \mathbf{P}_x ; see Theorem 2 in [10] and Definition 2 in [6]. Note that, the special case $\theta = 0$ of our model coincides with *homogeneous* growth-fragmentations (self-similar with $\alpha = 0$); however, the OU type scaling property with $\theta \neq 0$ does not have an analogue in (pure) fragmentations.

The first main purpose of this work is to provide a construction of such OU type growth-fragmentation processes. Our approach is based on the idea introduced by Bertoin [9] to build homogeneous growth-fragmentations (which he called *compensated fragmentation processes*). Specifically, the starting point of his approach is the observation (see also [13]) that upon a logarithmic transformation, homogeneous (pure) fragmentations can be viewed as continuous time branching random walks. Replacing branching random walks by *branching Lévy processes*, in which an atom also moves according to a Lévy process, and then taking an exponential transform, one obtains homogeneous growth-fragmentations. It is remarkable that a monotonicity argument is used in this approach such that the branching events are allowed to occur with an infinite intensity. See also [14] for a related construction of binary self-similar growth-fragmentations.

Similarly, we introduce certain *branching OU type processes*, in which an atom evolves as an OU type process, then the associated growth-fragmentations naturally fulfill the desired properties. We stress that in our model the branching rate can also be infinite. The technical difficulty in adopting this approach is that one needs to check that such a growth-fragmentation does not explode, that is, for every $x > 0$, only a finite number of fragments have size greater than x at any time. This is justified by Theorem 2.8.

In this paper we establish two major results on OU type growth-fragmentations. We first prove (Theorem 3.4) the convergence of a sequence of OU type growth-fragmentations when their characteristics converge in some sense. This conclusion generalizes Theorem 2 in [9]. The other result (Corollary 3.16) concerns the long-time asymptotic behavior. Roughly speaking, under certain conditions the average size of the particles converges to a stationary distribution. This law of large numbers should be compared with the limit theorems for empirical measures of self-similar fragmentations and growth-fragmentations [12, 19], as well as the law of large numbers in the context of branching Gaussian OU processes [22].

We also find that OU type growth-fragmentations bear a connection with Bertoin's *Markovian growth-fragmentations* [10] and that they are the stochastic counterparts of certain (deterministic) growth-fragmentation equations; see [15, 20, 21, 29] for related works on the latter topic.

The rest of this paper is organized as follows. In Section 2 we construct OU type growth-fragmentations and establish their regularities. In Section 3, we find growth-fragmentation equations related to our model, establish a convergence criterion of a sequence of OU type growth-fragmentations, and prove the law of large numbers. In Section 4 we discuss the connections between our model and Markovian growth-fragmentations [10]. Finally, we present a remarkable example related to a destruction process of infinite random recursive tree in Section 5.

2 Construction of OU type growth-fragmentation processes

In this section, we present the construction of OU type growth-fragmentation processes. We first recall some background on OU type processes and a connection between homogeneous fragmentations and branching random walks. Then we introduce branching OU type processes. Using the latter, we construct OU type growth-fragmentation processes and establish some fundamental properties.

2.1 Preliminaries: Ornstein-Uhlenbeck type processes

Let us present some fundamental background on Ornstein-Uhlenbeck (OU) type processes driven by Lévy processes; see [1] or Section 17 in [32]. We also refer to [7] for properties of Lévy processes. Implicitly, throughout this work we only consider OU type processes without positive jumps.

Let ξ be a Lévy process without positive jumps, possibly killed, which is often referred to as a **spectrally negative Lévy process**. It is characterized by its Laplace exponent $\Phi : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\mathbb{E} \left[e^{q\xi(t)} \right] = e^{\Phi(q)t}, \quad \text{for all } t, q \geq 0.$$

The function Φ is continuous and convex on $[0, \infty)$. Further, it is given by the Lévy-Khintchine formula

$$\Phi(q) = -k + \frac{1}{2}\sigma^2 q^2 + cq + \int_{(-\infty, 0)} (e^{qy} - 1 + q(1 - e^y))\Lambda(dy), \quad q \geq 0, \quad (2.1)$$

where $k \geq 0$, $\sigma \geq 0$, $c \in \mathbb{R}$, and the Lévy measure Λ on $(-\infty, 0)$ satisfies

$$\int_{(-\infty, 0)} (|y|^2 \wedge 1)\Lambda(dy) < \infty. \quad (2.2)$$

We say ξ has characteristics (σ, c, Λ, k) . In the Lévy-Khintchine formula, we can also replace $q(1 - e^y)$ in the integral by $-qy\mathbb{1}_{\{y > -1\}}$, as often in the literature, then we need to change the drift coefficient c .

Let $\theta \in \mathbb{R}$, we next define an **Ornstein-Uhlenbeck (OU) type process** Z with characteristics $(\sigma, c, \Lambda, k, \theta)$ or simply (Φ, θ) , starting from $Z(0) = z \in \mathbb{R}$, by

$$Z(t) = e^{-\theta t}z + \int_0^t e^{-\theta(t-s)}d\xi(s), \quad t \geq 0. \quad (2.3)$$

By convention, if ξ is killed at time $\zeta \geq 0$, then $Z(t) := -\infty$ for every $t \geq \zeta$. When $\theta > 0$, Z is called an *inward* OU type process; respectively, while $\theta < 0$, Z is called an *outward* OU type process. Note that in the literature, OU type processes often only refer to the inward case ($\theta > 0$). Further, it is well-known that Z is

the pathwise unique solution of the stochastic integral equation

$$Z(t) = z + \xi(t) - \theta \int_0^t Z(s) ds,$$

and that there is

$$\mathbb{E} [\exp (q Z(t))] = \exp \left(e^{-\theta t} z q + \int_0^t \Phi(q e^{-\theta s}) ds \right), \quad \text{for all } t, q \geq 0; \quad (2.4)$$

see (17.2) and Lemma 17.1 in [32].

Under certain conditions, an inward OU type process converges in distribution to its stationary distribution.

Lemma 2.1 (Theorem 17.5 and 17.11 in [32]). *If $\theta > 0$ and Λ satisfies*

$$\int_{(-\infty, -\log 2)} \log |y| \Lambda(dy) < \infty, \quad (2.5)$$

then the OU type process Z possesses a unique stationary distribution Π , which is a probability measure on \mathbb{R} with Laplace transform

$$\int_{\mathbb{R}} e^{qy} \Pi(dy) = \exp \left(\int_0^\infty \Phi(e^{-\theta s} q) ds \right), \quad q \geq 0.$$

Further, for every bounded and continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ there is

$$\lim_{t \rightarrow \infty} \mathbb{E}[g(Z(t))] = \int_{\mathbb{R}} g(y) \Pi(dy).$$

If (2.5) does not hold, then Z does not have any stationary distribution.

We remark that the stationary distribution Π is *self-decomposable*, which means that if a random variable Y has law Π , then for every constant $r \in (0, 1)$, there exists an independent random variable $Y^{(r)}$, such that $Y \stackrel{d}{=} rY + Y^{(r)}$. Conversely, every self-decomposable measure is the stationary distribution of a certain OU type process. See Definition 15.1 and Theorem 17.5 in [32] for details.

2.2 Homogeneous fragmentation processes

We present in this section a connection between homogeneous fragmentations and branching random walks, which was developed in Section 2 in [9]. This will help us to understand the construction of OU type growth-fragmentations.

Denote the space of mass-partitions by

$$\mathcal{S} := \left\{ \mathbf{s} := (s_1, s_2, \dots) : s_1 \geq s_2 \geq \dots \geq 0, \sum_{i=1}^\infty s_i \leq 1 \right\}.$$

A **homogeneous fragmentation \mathbf{X}** with no erosion is characterized by a *dislocation measure* ν on \mathcal{S} . Suppose that ν is **finite** (this constraint is needed only in this subsection and will be released later on), then \mathbf{X} describes the following particle system. Initially, there is a single particle with mass 1. Each particle of mass $x > 0$ splits at rate $\nu(\mathcal{S})$, and generates a sequence of particles with masses (xs_1, xs_2, \dots) , where $(s_1, s_2, \dots) \in \mathcal{S}$ has distribution $\nu(\cdot)/\nu(\mathcal{S})$. Each child fragment continues in a similar way.

Let us give a formal construction of \mathbf{X} via a certain (continuous time) branching random walk \mathcal{Z} defined as follows. We first introduce some notation. The *Ulam-Harris tree* is $\mathbb{U} := \bigcup_{n=0}^\infty \mathbb{N}^n$ with $\mathbb{N} := \{1, 2, 3, \dots\}$ and

$\mathbb{N}^0 := \{\emptyset\}$ by convention. So an element $u \in \mathbb{U}$ is a finite sequence of natural numbers $u = (n_1, \dots, n_{|u|})$, where $|u| \in \mathbb{N}$ stands for the generation of u . Write $u_- = (n_1, \dots, n_{|u|-1})$ for her mother and $uk = (n_1, \dots, n_{|u|}, k)$ for her k -th daughter with $k \in \mathbb{N}$. With convention $e^{-\infty} := 0$, we also introduce the space

$$\mathcal{R} := \left\{ \mathbf{r} = (r_1, r_2, \dots) : 0 \geq r_1 \geq r_2 \geq \dots \geq -\infty, \sum_{i=1}^{\infty} e^{r_i} \leq 1 \right\}.$$

Proposition 2.2 ([9]). *Let ν be a finite measure on \mathcal{S} and μ be the image of ν by the map $(s_1, s_2, \dots) \mapsto (\log s_1, \log s_2, \dots)$, with convention $\log 0 := -\infty$, then μ is a finite measure on \mathcal{R} . Let $(\lambda_u, u \in \mathbb{U})$ be a family of i.i.d. exponential random variables with parameter $\mu(\mathcal{R})$, $((\Delta a_{ui})_{i \in \mathbb{N}}, u \in \mathbb{U})$ be a family of i.i.d. random variables with distribution $\mu(\cdot)/\mu(\mathcal{R})$. The two families are independent. With initial values $b_\emptyset = 0$ and $a_\emptyset = 0$, we define recursively*

$$a_{ui} = a_u + \Delta a_{ui}, \quad b_{ui} = b_u + \lambda_u, \quad \text{for every } u \in \mathbb{U}, i \in \mathbb{N}.$$

For every $u \in \mathbb{U}$ the triple (a_u, b_u, λ_u) stands for the position, the birth time and the lifetime respectively of the particle indexed by u . For every $t \geq 0$, denote the positions of particles alive at time t by the multiset (which is like a set but allows multiple instances of elements)

$$\mathcal{Z}(t) := \{a_u \in \mathbb{R} : u \in \mathbb{U}, t \in [b_u, b_u + \lambda_u)\}.$$

Let $\mathbf{X}(t) := (X_1(t), X_2(t), \dots)$ be the null-sequence obtained by listing the elements of $\{e^z, z \in \mathcal{Z}(t)\}$ in decreasing order. Then the process \mathbf{X} is a homogeneous fragmentation with no erosion and finite dislocation measure ν .

Note that if $\Delta a_{ui} = -\infty$, then by convention $a_{ui} := -\infty$, which means that the atom ui (as well as its descendants) is not taken into account.

We now introduce a different representation of the branching random walk \mathcal{Z} . The key point is that we now distinguish between two types of branching events, namely, those in which exactly one particle is generated, which corresponds to those $u \in \mathbb{U}$ such that $(\Delta a_{u1}, \Delta a_{u2}, \dots)$ is included in

$$\mathcal{R}_1 := \{\mathbf{r} \in \mathcal{R} : r_1 > \infty, r_2 = r_3 = \dots = -\infty\},$$

and the others (those correspond to $\mathcal{R} \setminus \mathcal{R}_1$; note that $(-\infty, -\infty, \dots) \in \mathcal{R} \setminus \mathcal{R}_1$, which means a particle is killed). We shall next treat the former as displacements of atoms, but not as branching events, and thus change accordingly the genealogy of the branching random walk. From this point of view, we have the following description.

Proposition 2.3 ([9]). *Let ν be a finite measure on \mathcal{S} and μ be the image of ν by the map $(s_1, s_2, \dots) \mapsto (\log s_1, \log s_2, \dots)$. Let $(\lambda_u, u \in \mathbb{U})$ be a family of i.i.d. exponential random variables with parameter $\mu(\mathcal{R} \setminus \mathcal{R}_1)$, $((\Delta a_{ui})_{i \in \mathbb{N}}, u \in \mathbb{U})$ be a family of i.i.d. random variables with distribution $\mu(\cdot \mid \mathcal{R} \setminus \mathcal{R}_1)$, and $(\xi_u, u \in \mathbb{U})$ be a family of i.i.d. compound Poisson processes with Lévy measure given by the image of the restriction of $\mu|_{\mathcal{R}_1}$ via the map $\mathbf{r} \rightarrow r_1$ from \mathcal{R}_1 to $(-\infty, 0)$. With initial values $b_\emptyset = 0$ and $a_\emptyset = 0$, we define recursively*

$$a_{ui} = a_u + \Delta a_{ui} + \xi_u(\lambda_u), \quad b_{ui} = b_u + \lambda_u, \quad \text{for every } u \in \mathbb{U}, i \in \mathbb{N}.$$

For every $u \in \mathbb{U}$ the triple (a_u, b_u, λ_u) stands for the position, the birth time and the lifetime respectively of the particle indexed by u . For every $t \geq 0$, define a multiset

$$\tilde{\mathcal{Z}}(t) := \{a_u + \xi_u(t - b_u) : u \in \mathbb{U}, t \in [b_u, b_u + \lambda_u)\}$$

by the positions of particles alive at time t , and let $\mathbf{X}(t) := (X_1(t), X_2(t), \dots)$ be the null-sequence obtained by listing the elements of $\{\exp(z), z \in \tilde{Z}(t)\}$ in decreasing order. Then the process \mathbf{X} is a homogeneous fragmentation with no erosion and finite dislocation measure ν .

2.3 OU type branching Markov chains

We now extend the construction of \tilde{Z} as in Proposition 2.3 to *OU type branching Markov chains*, then intuitively we derive OU type growth-fragmentations from such systems by an exponential transform.

Recall that in Proposition 2.3, the movement of an atom is a compound Poisson process ξ with Lévy measure Λ_1 , the image of the restriction of $\mu|_{\mathcal{R}_1}$ via the map $\mathbf{r} \rightarrow r_1$ from \mathcal{R}_1 to $(-\infty, 0)$. Here ξ is naturally replaced by any OU type process Z (without killing) with the same Lévy measure Λ_1 . The splitting mechanism is still given by $\mu|_{\mathcal{R} \setminus \mathcal{R}_1}$, such that a particle at any position $y \in \mathbb{R}$ splits into two or more particles at $y + \mathbf{r}$ with rate $\mu|_{\mathcal{R} \setminus \mathcal{R}_1}(d\mathbf{r})$; the particle born at position $y + r_i$ evolves according to the law of Z with $Z(0) = y + r_i$.

Further, we observe that to define these dynamics, we do not need μ to be finite. We therefore release this constraint but only suppose that μ is a sigma-finite measure on \mathcal{R} that satisfies

$$\int_{\mathcal{R}} (1 - e^{r_1})^2 \mu(d\mathbf{r}) < \infty, \quad (2.6)$$

and that $\mu(\mathcal{R} \setminus \mathcal{R}_1) < \infty$. Then Λ_1 is still a Lévy measure (that satisfies (2.2)) such that Z is well-defined, and this particle system can be rigorously constructed in the following way.

Definition 2.4. Let $\theta \in \mathbb{R}$, $\sigma \geq 0$, $c \in \mathbb{R}$, and μ be a sigma-finite measure in \mathcal{R} such that (2.6) holds and $\mu(\mathcal{R} \setminus \mathcal{R}_1) < \infty$. Consider three independent families $(\lambda_u)_{u \in \mathbb{U}}$, $(Z_u)_{u \in \mathbb{U}}$ and $(\Delta a_{ui}, i \in \mathbb{N})_{u \in \mathbb{U}}$:

- $(\lambda_u)_{u \in \mathbb{U}}$ is a family of i.i.d. exponential variables with parameter $\mu(\mathcal{R} \setminus \mathcal{R}_1)$.
- $(Z_u)_{u \in \mathbb{U}}$ is a family of i.i.d. OU type processes, starting from $Z_u(0) = 0$, with characteristics (ψ, θ) , where

$$\psi(q) := \frac{1}{2} \sigma^2 q^2 + \left(c + \int_{\mathcal{R} \setminus \mathcal{R}_1} (1 - e^{r_1}) \mu(d\mathbf{r}) \right) q + \int_{\mathcal{R}_1} (e^{qr_1} - 1 + q(1 - e^{r_1})) \mu(d\mathbf{r}), \quad q \geq 0. \quad (2.7)$$

- $(\Delta a_{ui}, i \in \mathbb{N})_{u \in \mathbb{U}}$ is a family of i.i.d. sequences, each sequence being distributed according to the conditional probability $\mu(\cdot \mid \mathcal{R} \setminus \mathcal{R}_1)$.

With initial values $b_\emptyset = 0$ and $a_\emptyset = 0$, we define recursively

$$a_{ui} := e^{-\theta \lambda_u} a_u + Z_u(\lambda_u) + \Delta a_{ui}, \quad b_{ui} := b_u + \lambda_u, \quad \text{for every } u \in \mathbb{U}, i \in \mathbb{N}.$$

For every $u \in \mathbb{U}$ the triple (a_u, b_u, λ_u) stands for the position at birth, the birth time and the lifetime respectively of the particle indexed by u . This particle moves according to $(e^{-\theta r} a_u + Z_u(r))_{r \geq 0}$, which has the law of Z with $Z(0) = a_u$ by (2.3). Then the positions of the particles alive at time $t \geq 0$ form a multiset

$$\mathcal{Z}(t) := \{e^{-\theta(t-b_u)} a_u + Z_u(t-b_u) : u \in \mathbb{U}, b_u \leq t < b_u + \lambda_u\}, \quad t \geq 0.$$

The process \mathcal{Z} is called an **OU type branching Markov chain** with characteristics (σ, c, μ, θ) .

The choice of the drift coefficient in (2.7) is for the following purposes. First, this is consistent with Definition 1 in [9]. So for the case $\theta = 0$, an OU type branching Markov chain with characteristics $(\sigma^2, c, \mu, 0)$ is a *branching Lévy process* with characteristics (σ^2, c, μ) . Second, this enables us to obtain an important embedding property that we shall now present. For each $\ell \geq 0$, we cut an OU type branching Markov chain \mathcal{Z}

with characteristics (σ, c, μ, θ) at level ℓ , by keeping at each dislocation the child particle which is the closest to the parent, and by suppressing the other child particles if and only if its distance to the position of the parent at death is larger than or equal to ℓ . Let $B(\ell) \subset \mathbb{U}$ be the set of individuals that are killed by this cutting operation, so $u = (u_1, \dots, u_{|u|}) \in B(\ell)$ if and only if

$$\Delta a_{u_1, \dots, u_j} \leq -\ell \text{ and } u_j \geq 2 \text{ for some } j = 1, \dots, |u|.$$

For every $r \in [-\infty, 0]$, set

$$r^{(\ell)} := \begin{cases} r & \text{if } r > -\ell, \\ -\infty & \text{otherwise.} \end{cases}$$

Then for every $\mathbf{r} = (r_1, r_2, r_3, \dots) \in \mathcal{R}$, we define

$$\mathbf{r}^{(\ell)} := (r_1, r_2^{(\ell)}, r_3^{(\ell)}, \dots). \quad (2.8)$$

Let $\mu^{(\ell)}$ be the image of μ by the map $\mathbf{r} \mapsto \mathbf{r}^{(\ell)}$.

Lemma 2.5 (Key embedding property). *The truncated process*

$$\mathcal{Z}^{(\ell)}(t) := \{e^{-\theta(t-b_u)} a_u + Z_u(t - b_u) : u \in \mathbb{U}, u \notin B(\ell), b_u \leq t < b_u + \lambda_u\}, \quad t \geq 0 \quad (2.9)$$

is an OU type branching Markov chain with characteristics $(\sigma, c, \mu^{(\ell)}, \theta)$.

See Lemma 3 in [9] for an analogous result for branching Lévy processes.

As a particular case of the truncation, when $\ell = 0$, at each branching event we only keep the child which is the closest to the parent, and discard all the others. Therefore, at each time $t \geq 0$ it remains only one particle, called the *selected atom*. Formally, with the notation of Definition 2.4, the position of the selected atom is given by

$$Z_*(t) := e^{-\theta(t-b_{\bar{1}_n})} a_{\bar{1}_n} + Z_{\bar{1}_n}(t - b_{\bar{1}_n}), \quad t \in [b_{\bar{1}_n}, b_{\bar{1}_n} + \lambda_{\bar{1}_n}),$$

where $\bar{1}_n := (1, 1, \dots, 1) \in \mathbb{N}^n$ for every $n \geq 0$.

Lemma 2.6. *The position of the selected atom Z_* is an OU type process with characteristics $(\sigma, c, \Lambda_*, 0, \theta)$, where Λ_* be the image of μ via the map $\mathbf{r} \rightarrow r_1$ from \mathcal{R} to $(-\infty, 0)$. Equivalently, Z_* has characteristics (Φ_*, θ) , where*

$$\Phi_*(q) = \frac{1}{2} \sigma^2 q^2 + cq + \int_{\mathcal{R}} (e^{qr_1} - 1 + q(1 - e^{r_1})) \mu(d\mathbf{r}), \quad q \geq 0.$$

The proof of Lemma 2.5 and Lemma 2.6 are deferred to Section 2.6.

2.4 OU type branching Markov processes

We next extend OU type branching Markov chains to a more general class of *OU type branching Markov processes*, such that the branching rate could be infinite. Along the lines of Definition 2 in [9], our approach relies on the key embedding property, Lemma 2.5, which enables us to consider increasing limits.

Specifically, we release the assumption that $\mu(\mathcal{R} \setminus \mathcal{R}_1) < \infty$ but only suppose that (2.6) holds. For every $\ell \geq 0$, write $\mu^{(\ell)}$ for the image of μ by the map $\mathbf{r} \mapsto \mathbf{r}^{(\ell)}$, then

$$\mu^{(\ell)}(\mathcal{R} \setminus \mathcal{R}_1) = \mu(\mathbf{r}^{(\ell)} \notin \mathcal{R}_1) = \mu(r_1 = -\infty \text{ or } r_2 > -\ell) \leq \mu(1 - e^{r_1} > e^{-\ell}) < \infty.$$

By Lemma 2.5 and Kolmogorov's extension theorem, we can build a family of processes on the same probability space, which we still denote by $(\mathcal{Z}^{(\ell)})_{\ell \geq 0}$, such that each $\mathcal{Z}^{(\ell)}$ is an OU type branching Markov chain with

characteristics $(\sigma, c, \mu^{(\ell)}, \theta)$, and

$$(\mathcal{Z}^{(\ell)})^{(\ell')} = \mathcal{Z}^{(\ell')} \quad \text{for every } \ell' > \ell,$$

where $(\mathcal{Z}^{(\ell)})^{(\ell')}$ denotes the process obtained by cutting $\mathcal{Z}^{(\ell)}$ at level ℓ' .

Definition 2.7. Suppose that (2.6) holds. In the notation above, we define (by the increasing limit)

$$\mathcal{Z}(t) := \lim_{\ell \rightarrow \infty} \uparrow \mathcal{Z}^{(\ell)}(t), \quad t \geq 0.$$

We call \mathcal{Z} an **OU type branching (Markov) process** with characteristics (σ, c, μ, θ) .

2.5 OU type growth-fragmentation processes

We finally construct OU type growth-fragmentation processes. Let $\sigma \geq 0$, $c \in \mathbb{R}$, $\theta \in \mathbb{R}$ and ν be a sigma-finite measure on the space of mass-partitions \mathcal{S} , which satisfies

$$\int_{\mathcal{S}} (1 - s_1)^2 \nu(ds) < \infty. \quad (2.10)$$

Introduce the **cumulant** $\kappa : [0, \infty) \rightarrow (-\infty, \infty]$, that plays an important role in this work (and also for compensated fragmentations [9]):

$$\kappa(q) := \frac{1}{2} \sigma^2 q^2 + cq + \int_{\mathcal{S}} \left(\sum_{i=1}^{\infty} s_i^q - 1 + q(1 - s_1) \right) \nu(ds), \quad q \geq 0, \quad (2.11)$$

with the convention that $0^0 := 0$. Notice that

$$\kappa(q) = \Phi_*(q) + \int_{\mathcal{S}} \sum_{i=2}^{\infty} s_i^q \nu(ds), \quad (2.12)$$

where

$$\Phi_*(q) := \frac{1}{2} \sigma^2 q^2 + cq + \int_{\mathcal{S}} (s_1^q - 1 + q(1 - s_1)) \nu(ds), \quad q \geq 0. \quad (2.13)$$

It is known that, when (2.10) holds, Φ_* is the Laplace exponent of a certain spectrally negative Lévy process; see Corollary 1 in [9]. In particular, Φ_* is finite and continuous on $[0, \infty)$. Denote

$$\text{dom}(\kappa) := \{q \geq 0 : \kappa(q) < \infty\},$$

then we observe from (2.12) that

$$q \in \text{dom}(\kappa) \text{ if and only if } \int_{\mathcal{S}} \sum_{i=2}^{\infty} s_i^q \nu(ds) < \infty.$$

Further, κ is continuous and convex in $\text{dom}(\kappa)$. As $\sum_{i=2}^{\infty} s_i^q \leq (1 - s_1)^q$ for $q \geq 2$, condition (2.10) ensures that $[2, \infty) \subset \text{dom}(\kappa)$.

Write μ for the image of ν by the map $(s_1, s_2, \dots) \mapsto (\log(s_1), \log(s_2), \dots) \in \mathcal{R}$, then μ is a sigma-finite measure on \mathcal{R} , and (2.10) ensures that μ satisfies (2.6). Hence we are allowed to construct by Definition 2.7 an OU type branching Markov process \mathcal{Z} with characteristics (σ, c, μ, θ) . Recall that c_o^\downarrow is the space of all decreasing null sequences endowed with the ℓ^∞ -distance, i.e. $\|\mathbf{x} - \mathbf{y}\|_\infty = \sup_{i \in \mathbb{N}} |x_i - y_i|$ for $\mathbf{x} = (x_1, x_2, \dots) \in c_o^\downarrow$ and $\mathbf{y} = (y_1, y_2, \dots) \in c_o^\downarrow$.

Theorem 2.8. *For every $t \geq 0$, the elements of $\{\exp(z) : z \in \mathcal{Z}(t)\}$ can be rearranged in a decreasing null sequence*

$$\mathbf{X}(t) := (X_1(t), X_2(t), \dots) \in c_o^\downarrow.$$

Further, for every $\alpha \in \text{dom}(\kappa)$ and $q \geq \alpha(1 \vee e^{\theta t})$, we have

$$\mathbb{E} \left[\sum_{i=1}^{\infty} X_i(t)^q \right] = \exp \left(\int_0^t \kappa(qe^{-\theta s}) ds \right). \quad (2.14)$$

The proof of Theorem 2.8 is postponed to Section 2.6.

Definition 2.9. *With the notation of Theorem 2.8, the process $\mathbf{X} := (\mathbf{X}(t), t \geq 0)$ is called an **OU type growth-fragmentation process** with characteristics (σ, c, ν, θ) .*

Remark 2.10. *When $\theta = 0$, an OU type growth-fragmentation with characteristics $(\sigma, c, \nu, 0)$ is a compensated fragmentation with characteristics (σ, c, ν) in the sense of Definition 3 in [9]. To avoid duplication, this case will be implicitly excluded hereafter.*

Roughly speaking, $\sigma \geq 0$ describes the fluctuations of the size, the constant $c \in \mathbb{R}$ represents the deterministic dilation (resp. erosion) coefficient when $c > 0$ (resp. $c < 0$). The measure ν is called the *dislocation measure*. For every $\mathbf{s} \in \mathcal{S}$, a fragment of size $x > 0$ splits into a sequence of fragments $x\mathbf{s}$ at rate $\nu(d\mathbf{s})$. The constant $\theta \in \mathbb{R}$ characterizes the speed at which the size of a fragment evolves towards (when $\theta > 0$) or away from (when $\theta < 0$) the value 1 (as the central location of an OU type process is 0). Due to Definition 2.4 and 2.7, an OU type growth-fragmentation \mathbf{X} is always assumed (without loss of generality) to start from one fragment of unit size, i.e. $\mathbf{X}(0) := (1, 0, 0, \dots)$, unless otherwise specified.

Recall from Definition 2.7 that the OU type branching Markov process \mathcal{Z} is the increasing limit of a family of OU type branching Markov chains $(\mathcal{Z}^{(\ell)}, \ell \geq 0)$. We thus define for every $\ell \geq 0$ a *truncated* OU type growth-fragmentation $\mathbf{X}^{(\ell)}$ by the exponential of $\mathcal{Z}^{(\ell)}$ (rearranged in decreasing order). In particular when $\ell = 0$, in the truncated system $\mathbf{X}^{(0)}$ there is always at most one fragment, called *the selected fragment* of \mathbf{X} . We stress that it is not necessarily the largest one in the system.

Lemma 2.11 (Selected fragment). *The size of the selected fragment $(X_*(t), t \geq 0)$ is the exponential of an OU type process with characteristics (Φ_*, θ) , with Φ_* given by (2.13).*

Proof. The law of $\log X_*$ is given by Lemma 2.6. □

With the help of Theorem 2.8, we shall establish some fundamental properties of \mathbf{X} in the rest of this section. We first prove that \mathbf{X} is a time-homogeneous Markov process. In this direction, let us define a family of probability measures. Specifically, let $\alpha \in \text{dom}(\kappa)$ and $\mathbf{x} = (x_1, x_2, \dots) \in \ell^{\alpha\downarrow} \subset c_o^\downarrow$, where $\ell^{\alpha\downarrow}$ denotes the space of decreasing null sequences with finite ℓ^α -norm, i.e. $\|\mathbf{x}\|_{\ell^\alpha} := (\sum_{i=1}^{\infty} |x_i|^\alpha)^{\frac{1}{\alpha}} < \infty$. Let $(\mathbf{X}^{[j]}, j \in \mathbb{N})$ be a sequence of i.i.d. copies of \mathbf{X} . We have for every $t \geq 0$ and $q \geq \alpha(e^{\theta t} \vee 1)$ that

$$\mathbb{E} \left[\sum_{j \geq 1} \sum_{i \geq 1} \left| x_j^{e^{-\theta t}} X_i^{[j]}(t) \right|^q \right] = \exp \left(- \int_0^t \kappa(qe^{-\theta s}) ds \right) \sum_{j \geq 1} |x_j|^{qe^{-\theta t}} < \infty,$$

so the elements (repeated according to their multiplicity) of $\{x_j^{e^{-\theta t}} X_i^{[j]}(t), i, j \in \mathbb{N}\}$ can be rearranged in decreasing order. Write $\mathbf{P}_{\mathbf{X}}$ for the law of the resulting process on c_o^\downarrow .

Proposition 2.12 (Markov property). *Let $s \geq 0$ and suppose that $\mathbf{X}(s) \in \ell^{\alpha\downarrow}$ for $\alpha \in \text{dom}(\kappa)$. Then the conditional distribution of the process $(\mathbf{X}(t+s), t \geq 0)$ given $(\mathbf{X}(r), 0 \leq r \leq s)$ is $\mathbf{P}_{\mathbf{X}(s)}$.*

This statement clearly ensures that \mathbf{X} fulfills the properties **(P1)** and **(P2)** in the introduction.

Proof of Proposition 2.12. For every $\ell \geq 0$, consider the *truncated* OU type growth-fragmentation $\mathbf{X}^{(\ell)}$ and the corresponding OU type branching Markov chain $\mathcal{Z}^{(\ell)}$. It is plain from Definition 2.4 that $\mathbf{X}^{(\ell)}$ fulfills the claimed Markov property. This observation and Theorem 2.8 entail that the Markov property also holds for \mathbf{X} . See the proof of Proposition 2 in [10] for similar arguments and we omit the details. \square

Combining Theorem 2.8 and Proposition 2.12, we immediately obtain the following non-negative martingales, which should be compared with the famous *additive martingales* in the context of (pure) fragmentations [13] or branching random walks [16].

Proposition 2.13 (Additive martingales). *Let \mathbf{X} be an OU type growth-fragmentation with cumulant κ .*

(i) *If $\theta < 0$, then for every $q \in \text{dom}(\kappa)$, the process*

$$\left(\exp \left(- \int_0^t \kappa(qe^{-\theta s}) ds \right) \sum_{i=1}^{\infty} X_i(t)^q, \quad t \geq 0 \right) \quad \text{is a martingale.}$$

(ii) *If $\theta > 0$, then for every $\alpha \in \text{dom}(\kappa)$, the process*

$$\left(\exp \left(- \int_0^t \kappa(\alpha e^{\theta s}) ds \right) \sum_{i=1}^{\infty} X_i(t)^{\alpha e^{\theta t}}, \quad t \geq 0 \right) \quad \text{is a martingale.}$$

Proposition 2.14 (Feller-type property). *Let $\alpha \in \text{dom}(\kappa)$ and suppose that a sequence $\mathbf{x}_n \rightarrow \mathbf{x}_\infty$ in $\ell^{\alpha\downarrow}$. Then for every $t \geq 0$, there is the weak convergence*

$$(\mathbf{P}_{\mathbf{x}_n}(s), s \in [0, t]) \xrightarrow[n \rightarrow \infty]{} (\mathbf{P}_{\mathbf{x}_\infty}(s), s \in [0, t])$$

in the sense of finite dimensional distributions on $\ell^{q\downarrow}$ for every $q \geq \max(\alpha(e^{\theta t} \vee 1), 1)$.

Proof of Proposition 2.14. Similarly as in the proof of Corollary 2 in [9], we consider a sequence $(\mathbf{X}^{[j]}, j \in \mathbb{N})$ of i.i.d. copies of \mathbf{X} . As $q \geq \alpha(e^{\theta t} \vee 1)$, it follows from Theorem 2.8 that

$$\mathbb{E} \left[\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left| (x_{n,j}^{e^{-\theta t}} - x_{\infty,j}^{e^{-\theta t}}) X_i^{[j]}(t) \right|^q \right] = \exp \left(- \int_0^t \kappa(qe^{-\theta s}) ds \right) \sum_{j=1}^{\infty} |x_{n,j}^{e^{-\theta t}} - x_{\infty,j}^{e^{-\theta t}}|^q. \quad (2.15)$$

But then different estimations are needed for our case. More precisely, if $\theta > 0$, as the function $x \mapsto x^{e^{-\theta t}}$ is concave, then for every $j \geq 1$ there is

$$|x_{n,j}^{e^{-\theta t}} - x_{\infty,j}^{e^{-\theta t}}| \leq |x_{n,j} - x_{\infty,j}|^{e^{-\theta t}}.$$

We next consider the case $\theta < 0$. Since $\mathbf{x}_n \rightarrow \mathbf{x}_\infty$ in $\ell^{\alpha\downarrow}$, we may assume that for every $n \geq 1$, there is $|x_{n,j} - x_{\infty,j}| < 1$ for every $j \geq 1$, so $\|\mathbf{x}_n\|_{\ell^\infty} \leq \|\mathbf{x}_\infty\|_{\ell^\infty} + 1$. Therefore, with a constant $C(t) := e^{-\theta t}(\|\mathbf{x}_\infty\|_{\ell^\infty} + 1)^{e^{-\theta t}-1}$, we have

$$|x_{n,j}^{e^{-\theta t}} - x_{\infty,j}^{e^{-\theta t}}| \leq C(t) |x_{n,j} - x_{\infty,j}|, \quad \text{for every } j \in \mathbb{N}.$$

Combining these observations and that $\mathbf{x}_n \rightarrow \mathbf{x}_\infty$ in $\ell^{\alpha\downarrow}$, we deduce from (2.15) that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left| (x_{n,j}^{e^{-\theta t}} - x_{\infty,j}^{e^{-\theta t}}) X_i^{[j]}(t) \right|^q \right] = 0.$$

Write \mathbf{x}^\downarrow and \mathbf{y}^\downarrow for the decreasing rearrangements of two sequences \mathbf{x} and \mathbf{y} in ℓ^q . As the function $x \mapsto x^q$ is convex for $q \geq 1$, it follows from Theorem 3.5 in [28] that $\|\mathbf{x}^\downarrow - \mathbf{y}^\downarrow\|_{\ell^q}^q \leq \|\mathbf{x} - \mathbf{y}\|_{\ell^q}^q$. As a consequence, there is

$$\left\| (x_{n,j}^{e^{-\theta t}} X_i^{[j]}(t))^\downarrow - (x_{\infty,j}^{e^{-\theta t}} X_i^{[j]}(t))^\downarrow \right\|_{\ell^q}^q \leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left| (x_{n,j}^{e^{-\theta t}} - x_{\infty,j}^{e^{-\theta t}}) X_i^{[j]}(t) \right|^q,$$

which leads to

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left\| (x_{n,j}^{e^{-\theta t}} X_i^{[j]}(t))^\downarrow - (x_{\infty,j}^{e^{-\theta t}} X_i^{[j]}(t))^\downarrow \right\|_{\ell^q} \right] = 0.$$

From the description of $\mathbf{P}_{\mathbf{x}_n}$ and $\mathbf{P}_{\mathbf{x}_\infty}$, we deduce the Feller-type property. \square

We finally establish the regularity of the path of \mathbf{X} .

Proposition 2.15 (Càdlàg path). *Let $\alpha \in \text{dom}(\kappa)$, $T \geq 0$ and $q \geq \max(\alpha(e^{\theta T} \vee 1), 1)$. Then the process $(\mathbf{X}(t), t \in [0, T])$ possesses a càdlàg version in $\ell^{q\downarrow}$.*

In particular, the process \mathbf{X} possesses a càdlàg version in c_o^\downarrow .

Proof. We follow the same arguments as in the proof of Proposition 2 in [9]. For every $\ell \geq 0$, let $\mathcal{Z}^{(\ell)}$ be the truncated OU type branching Markov chain and $\mathbf{X}^{(\ell)}$ be its associated growth-fragmentation, then it follows plainly from the construction that $(\mathbf{X}^{(\ell)}(t), t \in [0, T])$ is almost surely càdlàg in $\ell^{q\downarrow}$. Therefore, to complete the proof, it suffices to prove that

$$\lim_{\ell \rightarrow \infty} \sup_{0 \leq t \leq T} \|\mathbf{X}(t) - \mathbf{X}^{(\ell)}(t)\|_{\ell^q}^q = 0 \quad \text{in probability.} \quad (2.16)$$

As $q \geq 1$, we have an inequality from the proof of Lemma 5 in [9]:

$$\|\mathbf{X}(t) - \mathbf{X}^{(\ell)}(t)\|_{\ell^q}^q \leq \|\mathbf{X}(t)\|_{\ell^q}^q - \|\mathbf{X}^{(\ell)}(t)\|_{\ell^q}^q, \quad \text{for every } t \in [0, T]. \quad (2.17)$$

By this inequality and the fact that $\kappa \geq \kappa^{(\ell)}$, we deduce that

$$\sup_{0 \leq t \leq T} \|\mathbf{X}(t) - \mathbf{X}^{(\ell)}(t)\|_{\ell^q}^q \leq A \sup_{0 \leq t \leq T} |M(t) - M^{(\ell)}(t)| + B(\ell) \sup_{0 \leq t \leq T} M(t),$$

where $M(t) := \exp(-\int_0^t \kappa(qe^{-\theta r})dr) \|\mathbf{X}(t)\|_{\ell^q}^q$ and $M^{(\ell)}(t) := \exp(-\int_0^t \kappa^{(\ell)}(qe^{-\theta r})dr) \|\mathbf{X}^{(\ell)}(t)\|_{\ell^q}^q$ are two martingales, $A := \sup_{0 \leq t \leq T} \exp(\int_0^t \kappa(qe^{-\theta r})dr)$ is a finite constant, and

$$B(\ell) := \sup_{0 \leq t \leq T} \left(\exp\left(\int_0^t \kappa(qe^{-\theta r})dr\right) - \exp\left(\int_0^t \kappa^{(\ell)}(qe^{-\theta r})dr\right) \right) \xrightarrow{\ell \rightarrow \infty} 0.$$

We know by monotone convergence that $\lim_{\ell \rightarrow \infty} \uparrow \|\mathbf{X}^{(\ell)}(T)\|_{\ell^q}^q = \|\mathbf{X}(T)\|_{\ell^q}^q$. Since $q \geq \alpha(e^{\theta T} \vee 1)$, it follows from Theorem 2.8 that $\mathbb{E}[\|\mathbf{X}(T)\|_{\ell^q}^q] < \infty$. Then by dominated convergence we have

$$\lim_{\ell \rightarrow \infty} \mathbb{E} \left[|M(T) - M^{(\ell)}(T)| \right] = 0.$$

Using Doob's inequality leads to (2.16). We have completed the proof. \square

Remark 2.16. *As a consequence of the Feller-type property and the càdlàg path, we deduce that \mathbf{X} fulfills the strong Markov property by a standard argument (approximate a general stopping time by a decreasing sequence of simple stopping times, and the Markov property holds for simple stopping times).*

2.6 Proofs of Lemma 2.5, Lemma 2.6 and Theorem 2.8

In this section, we complete the proofs of Lemma 2.5, Lemma 2.6 and Theorem 2.8. The key idea is to use the following decomposition of OU type branching Markov chains, which is motivated from Lemma 2 in [9].

A decomposition of OU type branching Markov chains Let \mathcal{Z} be an OU type branching Markov chain \mathcal{Z} with characteristics (σ, c, μ, θ) , so $\mu(\mathcal{R} \setminus \mathcal{R}_1) < \infty$ and (2.6) holds. Then \mathcal{Z} is closely related to a particle system \mathcal{W} , which we shall call **a branching random walk with an attractor**, whose law is determined by θ and μ in the following way. The system \mathcal{W} is similar to a branching random walk, but with an attractor at position 0, which attracts (resp. repels) the particles if $\theta > 0$ (resp. $\theta < 0$). More precisely, for any particle born at position $y \in \mathbb{R}$, its position at its lifetime $t \geq 0$ is $e^{-\theta t}y$. This particle dies after an exponential time with parameter $\mu(\mathcal{R} \setminus \mathcal{R}_1)$ and splits into a cloud of particles scattered on \mathbb{R} , whose positions relative to the death point of their parent are distributed according to the conditional probability $\mu(\cdot \mid \mathcal{R} \setminus \mathcal{R}_1)$. Each child moves and reproduces in the same way, independently of one another. By this rule, we build a particle system with one initial particle located at 0, and denote the positions of the particles alive at time $t \geq 0$ by a multiset $\mathcal{W}(t)$.

We also notice that the process

$$\tilde{\mathcal{W}}(t) := e^{\theta t} \mathcal{W}(t), \quad t \geq 0$$

is a (continuous time) branching random walk in a time-inhomogeneous environment: each particle branches at rate $\mu(\mathcal{R} \setminus \mathcal{R}_1)$; if a branching happens at (global) time $t \geq 0$, then the relative locations of its children are distributed according to $e^{\theta t} \mathbf{r}$, where \mathbf{r} has distribution $\mu(\cdot \mid \mathcal{R} \setminus \mathcal{R}_1)$.

The marginal distribution of an OU type branching Markov chain \mathcal{Z} is the same as that of \mathcal{W} superposing i.i.d. OU type processes in the following sense.

Lemma 2.17. *Following the notation above, suppose that $\mu(\mathcal{R} \setminus \mathcal{R}_1) < \infty$. Fix a time $t \geq 0$ and write $\mathcal{W}(t) = \{\{W_i : i \in I\}\}$. Then there exists a family of real valued random variables $(\beta_i)_{i \in I}$ such that the multiset*

$$\{\{W_i + \beta_i, i \in I\}\}$$

has the same law as $\mathcal{Z}(t)$, and conditionally on $\mathcal{W}(t)$, each β_i has Laplace transform

$$\mathbb{E} \left[e^{q\beta_i} \mid \mathcal{W}(t) \right] = \exp \left(\int_0^t \psi(qe^{-\theta s}) ds \right), \quad q \geq 0,$$

where ψ is given by (2.7).

Proof. We use a similar argument as the proof of Lemma 2 in [9]. Let $\partial\mathbb{U}$ be the set of infinite sequences of positive integers. For every $\bar{u} = (u_1, u_2, \dots) \in \partial\mathbb{U}$ and $i \geq 0$, write $\bar{u}_i := (u_1, u_2, \dots, u_i) \in \mathbb{N}^i$ (with $\bar{u}_0 := \emptyset$ by convention). Recall that \mathcal{Z} is built by Definition 2.4. In that framework, define recursively a sequence $(\tilde{a}_{\bar{u}_j})_{j \geq 0}$ such that $\tilde{a}_{\bar{u}_0} := 0$ and

$$\tilde{a}_{\bar{u}_{j+1}} := e^{-\theta \lambda_{\bar{u}_j}} \tilde{a}_{\bar{u}_j} + Z_{\bar{u}_j}(\lambda_{\bar{u}_j}).$$

We further define a process $Z_{\bar{u}}$ by

$$Z_{\bar{u}}(t) := e^{-\theta(t-b_{\bar{u}_j})} \tilde{a}_{\bar{u}_j} + Z_{\bar{u}_j}(t - b_{\bar{u}_j}) \quad \text{for } t \in [b_{\bar{u}_j}, b_{\bar{u}_j} + \lambda_{\bar{u}_j}) \text{ with } j \geq 0. \quad (2.18)$$

Then it follows from the simple Markov property of OU type processes that each $Z_{\bar{u}}$ is an OU type process with characteristics (ψ, θ) . For every $\bar{u} \in \partial\mathbb{U}$, let $\eta_{\bar{u}}$ be a (time-inhomogeneous) compound Poisson process

which makes a jump of size $e^{\theta b_{\bar{u}_i}} \Delta a_{\bar{u}_i}$ at time $b_{\bar{u}_i}$ for every $i \geq 0$, i.e.

$$\eta_{\bar{u}}(t) := \sum_{i=0}^j \exp(\theta b_{\bar{u}_i}) \Delta a_{\bar{u}_i} \quad \text{for } t \in [b_{\bar{u}_j}, b_{\bar{u}_j} + \lambda_{\bar{u}_j}) \text{ with } j \geq 0.$$

We next equip the edges of \mathbb{U} with lengths, such that for every $u \in \mathbb{U}$ and $j \in \mathbb{N}$, the length of the edge connecting u and uj is λ_u , so the distance between each $u \in \mathbb{U}$ and the root \emptyset is b_u . Cutting the tree \mathbb{U} at height $t > 0$ (distance from the root) yields $L \subset \mathbb{U}$, i.e. $u \in L$ if and only if $b_u \leq t < b_u + \lambda_u$. Each $v \in L$ naturally corresponds to a subset $B_v \subset \partial\mathbb{U}$, that consists of all those $\bar{u} \in \partial\mathbb{U}$ stemming from v , and it is clear that the values $\eta_{\bar{u}}(t)$ (resp. $Z_{\bar{u}}(t)$) are the same for all $\bar{u} \in B_v$. So we define unambiguously

$$\eta_{B_v}(t) := \eta_{\bar{u}}(t) \quad \text{and} \quad Z_{B_v}(t) := Z_{\bar{u}}(t), \quad \bar{u} \in B_v.$$

We also observe that the family $(B_v, v \in L)$ are disjoint, forming a partition of $\partial\mathbb{U}$. Since for every $j \geq 0$ there is the identity

$$a_{\bar{u}_j} = \tilde{a}_{\bar{u}_j} + \sum_{i=0}^j \exp(-\theta(b_{\bar{u}_j} - b_{\bar{u}_i})) \Delta a_{\bar{u}_i} = \tilde{a}_{\bar{u}_j} + \exp(-\theta b_{\bar{u}_j}) \sum_{i=0}^j \exp(\theta b_{\bar{u}_i}) \Delta a_{\bar{u}_i},$$

then for every $t \geq 0$ we have the identity:

$$\mathcal{Z}(t) := \{\{e^{-\theta(t-b_u)} a_u + Z_u(t-b_u) : u \in L\} = \{\{e^{-\theta t} \eta_{B_v}(t) + Z_{B_v}(t) : v \in L\}.$$

Observing that $\{\{e^{-\theta t} \eta_{B_v}(t) : v \in L\}$ has the same law as $\mathcal{W}(t)$, we hence deduce the claim. \square

We next prove Lemma 2.5 and Lemma 2.6. The last ingredient is the next observation that plainly follows from (2.3).

Lemma 2.18. *If Z_1 and Z_2 are independent OU type processes with respective characteristics (Φ_1, θ) and (Φ_2, θ) , then $Z_1 + Z_2$ is an OU type process with characteristics $(\Phi_1 + \Phi_2, \theta)$.*

Proof of Lemma 2.5. The proof is an adaptation of the arguments of Lemma 3 in [9]. We shall check that $\mathcal{Z}^{(\ell)}$ fulfills Definition 2.4 with a different genealogy.

Consider $\bar{1} = (1, 1, 1, \dots) \in \partial\mathbb{U}$ and denote for every $i \in \mathbb{N}$ the ancestor of $\bar{1}$ in the i -th generation by $\bar{1}_i \in \mathbb{N}^i$, with $\bar{1}_0 = \emptyset$ by convention. With notation in Definition 2.4 and the proof of Lemma 2.17, we write $\mathbf{r}_i := \Delta a_{\bar{1}_i}$ for every $i \in \mathbb{N}$ and derive $\mathbf{r}_i^{(\ell)}$ from \mathbf{r}_i by (2.8). As \mathbf{r}_i has the law of $\mu(\cdot | \mathcal{R} \setminus \mathcal{R}_1)$, we easily deduce that $\mathbb{P}(\mathbf{r}_i^{(\ell)} \notin \mathcal{R}_1) = \frac{\mu^{(\ell)}(\mathcal{R} \setminus \mathcal{R}_1)}{\mu(\mathcal{R} \setminus \mathcal{R}_1)}$. Let $\bar{1}_N$ be the first node along the branch $\bar{1}$ such that $\mathbf{r}_N^{(\ell)} \notin \mathcal{R}_1$, then for all $i \leq N-1$ there is $\mathbf{r}_i^{(\ell)} \in \mathcal{R}_1$, which means that only the closest child of $\bar{1}_i$ is still alive in the system $\mathcal{Z}^{(\ell)}$ but the other children are all killed. Therefore, in the truncated system $\mathcal{Z}^{(\ell)}$ there is only one particle alive at any time before $a_{\bar{1}_N} + \lambda_{\bar{1}_N}$. We hence view the displacement of the only particle as the movement of the ancestor marked by \emptyset in the truncated system $\mathcal{Z}^{(\ell)}$, until its lifetime $\lambda_{\emptyset}^{(\ell)} := a_{\bar{1}_N} + \lambda_{\bar{1}_N}$, and then it splits into more than one particles, located relatively to the position of \emptyset at death by $\Delta a_{\emptyset}^{(\ell)} := \mathbf{r}_N^{(\ell)}$, which is a random variable of law $\mu^{(\ell)}(\cdot | \mathcal{R} \setminus \mathcal{R}_1)$. Since N has the geometric distribution with parameter $\frac{\mu^{(\ell)}(\mathcal{R} \setminus \mathcal{R}_1)}{\mu(\mathcal{R} \setminus \mathcal{R}_1)}$, from basic property of exponential random variable, we know that $\lambda_{\emptyset}^{(\ell)}$ has the exponential distribution with parameter $\mu(\mathcal{R} \setminus \mathcal{R}_1) \times \mathbb{P}(\mathbf{r}_i^{(\ell)} \notin \mathcal{R}_1) = \mu^{(\ell)}(\mathcal{R} \setminus \mathcal{R}_1)$.

We next investigate the distribution of the movement $Z_{\emptyset}^{(\ell)}$ of the ancestor \emptyset . By a similar discussion as in the proof of Lemma 2.17, the process $Z_{\emptyset}^{(\ell)}$ is the superposition of two independent OU type processes: one is

$Z_{\bar{1}}$ as in (2.18) with characteristics (ψ, θ) , and the other is an OU type process driven by (N, θ) , where N is a compound Poisson process on $(-\infty, 0)$ with Lévy measure

$$\mu(r_1 \in dz : \mathbf{r}^{(\ell)} \in \mathcal{R}_1, \mathbf{r} \notin \mathcal{R}_1), \quad z \in (-\infty, 0).$$

Therefore, we have by Lemma 2.18 that $Z_{\emptyset}^{(\ell)}$ is an OU type process with characteristics $(\psi^{(\ell)}, \theta)$ where

$$\psi^{(\ell)}(q) := \psi(q) + \int_{(-\infty, 0)} (e^{qz} - 1) \mu(r_1 \in dz : \mathbf{r}^{(\ell)} \in \mathcal{R}_1, \mathbf{r} \notin \mathcal{R}_1), \quad q \geq 0.$$

Using the fact that

$$\int_{\mathcal{R}} (1 - e^{r_1}) \mu^{(\ell)}(d\mathbf{r}) = \int_{\mathcal{R}} (1 - e^{r_1}) \mu(d\mathbf{r})$$

and that $\mathbf{r} \in \mathcal{R}_1$ implies $\mathbf{r}^{(\ell)} \in \mathcal{R}_1$, we deduce an identity

$$\psi^{(\ell)}(q) = \frac{1}{2} \sigma^2 q^2 + \left(c + \int_{\mathcal{R} \setminus \mathcal{R}_1} (1 - e^{r_1}) \mu^{(\ell)}(d\mathbf{r}) \right) q + \int_{\mathcal{R}_1} (e^{qr_1} - 1 + q(1 - e^{r_1})) \mu^{(\ell)}(d\mathbf{r}).$$

By iterating this argument and comparing with Definition 2.4, we complete that proof. \square

Proof of Lemma 2.6. Following the notation of the proof of Lemma 2.17, we consider the branch $\bar{1} = (1, 1, 1, \dots) \in \partial\mathbb{U}$, then the selected atom Z_* has the following decomposition:

$$Z_*(t) = Z_{\bar{1}}(t) + e^{-\theta t} \eta_{\bar{1}}(t), \quad t \geq 0.$$

The claim then follows from Lemma 2.18. \square

We finally turn to prove Theorem 2.8. To this vein, we need the following lemma. For every multiset $\pi := \{\{\pi_i, i \in I\}\}$ of elements in \mathbb{R} , we adopt the notation

$$\langle \pi, e^{qz} \rangle := \sum_{i \in I} e^{q\pi_i}, \quad q \geq 0.$$

Lemma 2.19. *Let \mathcal{W} be a branching random walk with an attractor. Then for every $t \geq 0$, we have*

$$\mathbb{E} [\langle \mathcal{W}(t), e^{qz} \rangle] = \exp \left(\int_0^t h(e^{-\theta s} q) ds \right), \quad q \geq 0,$$

where

$$h(q) := \int_{\mathcal{R} \setminus \mathcal{R}_1} \left(\sum_{i=1}^{\infty} e^{qr_i} - 1 \right) \mu(d\mathbf{r}).$$

Proof. Write τ_{\emptyset} for the branching time of the ancestor of system \mathcal{W} and $(\Delta a_i, i \in \mathbb{N})$ for the sequence of positions of the first generation at birth. It is plain from the construction that the sub-population generated by the particle at Δa_i has the same law as the process $(e^{-\theta t} \Delta a_i + \mathcal{W}(t))_{t \geq 0}$. Decompose at τ_{\emptyset} and use the

branching property, then there is

$$\begin{aligned}
m(q, t) &:= \mathbb{E} [\langle \mathcal{W}(t), e^{qz} \rangle] \\
&= \mathbb{E} [\langle \mathcal{W}(t), e^{qz} \rangle \mathbb{1}_{\{\tau_\emptyset > t\}}] + \mathbb{E} [\langle \mathcal{W}(t), e^{qz} \rangle \mathbb{1}_{\{\tau_\emptyset \leq t\}}] \\
&= \mathbb{P}(\tau_\emptyset > t) + \mathbb{E} \left[\mathbb{1}_{\{\tau_\emptyset \leq t\}} \sum_{i=1}^{\infty} \exp \left(q \Delta a_i e^{-\theta(t-\tau_\emptyset)} \right) \langle \mathcal{W}^i(t - \tau_\emptyset), e^{qz} \rangle \right] \\
&= e^{-\mu(\mathcal{R} \setminus \mathcal{R}_1)t} + \int_0^t \mu(\mathcal{R} \setminus \mathcal{R}_1) e^{-\mu(\mathcal{R} \setminus \mathcal{R}_1)s} m(q, t-s) ds \int_{\mathcal{R} \setminus \mathcal{R}_1} \sum_{i=1}^{\infty} \frac{\exp(q e^{-\theta(t-s)} r_i)}{\mu(\mathcal{R} \setminus \mathcal{R}_1)} \mu(d\mathbf{r}) \\
&= e^{-\mu(\mathcal{R} \setminus \mathcal{R}_1)t} + \int_0^t e^{-\mu(\mathcal{R} \setminus \mathcal{R}_1)s} m(q, t-s) \left(h(e^{-\theta(t-s)} q) + \mu(\mathcal{R} \setminus \mathcal{R}_1) \right) ds,
\end{aligned}$$

where $(\mathcal{W}^i, i \in \mathbb{N})$ are independent copies of \mathcal{W} , further independent of τ_\emptyset and $(\Delta a_i, i \in \mathbb{N})$. Changing variable in the integral by $t-s \mapsto s$, we have that

$$e^{\mu(\mathcal{R} \setminus \mathcal{R}_1)t} m(q, t) = 1 + \int_0^t e^{\mu(\mathcal{R} \setminus \mathcal{R}_1)s} m(q, s) (h(q e^{-\theta s}) + \mu(\mathcal{R} \setminus \mathcal{R}_1)) ds.$$

Solving this integral equation with initial condition $m(q, 0) = 1$, we obtain the desired identity. \square

Proof of Theorem 2.8. We first suppose that $\mu(\mathcal{R} \setminus \mathcal{R}_1) < \infty$, then it follows from Lemma 2.17 and Lemma 2.19 that for every $q \geq 0$ there is

$$\mathbb{E} [\langle \mathcal{Z}(t), e^{qz} \rangle] = \exp \left(\int_0^t \psi(e^{-\theta s} q) ds \right) \exp \left(\int_0^t h(e^{-\theta s} q) ds \right) = \exp \left(\int_0^t \kappa(e^{-\theta s} q) ds \right),$$

where κ is defined by (2.11) and plainly $\kappa = \psi + h$.

Now we consider $\mu(\mathcal{R} \setminus \mathcal{R}_1) = \infty$ (but μ fulfills (2.6)). For every $\ell \geq 0$, recall that the truncated process $\mathcal{Z}^{(\ell)}$ has characteristics $(\sigma, c, \mu^{(\ell)}, \theta)$, then it has cumulant

$$\kappa^{(\ell)}(q) = \frac{1}{2} \sigma^2 q^2 + cq + \int_{\mathcal{R}} \left(e^{q r_1} + \sum_{i=2}^{\infty} \mathbb{1}_{\{r_i > -\ell\}} e^{q r_i} - 1 + q(1 - e^{r_1}) \right) \mu(d\mathbf{r}), \quad q \geq 0.$$

Since μ fulfills (2.6), then we have $\mu^{(\ell)}(\mathcal{R} \setminus \mathcal{R}_1) < \infty$ and thus

$$\mathbb{E} [\langle \mathcal{Z}^{(\ell)}(t), e^{qz} \rangle] = \exp \left(\int_0^t \kappa^{(\ell)}(e^{-\theta s} q) ds \right), \quad q \geq 0.$$

Letting $\ell \rightarrow \infty$, it is plain that for every $p \geq \alpha$, there is

$$\lim_{\ell \rightarrow \infty} \uparrow \kappa^{(\ell)}(p) = \kappa(p) < \infty.$$

We hence deduce the claim by monotone convergence. \square

3 Properties of OU type growth-fragmentations

We continue to study OU type growth-fragmentations in this section. In Section 3.1 we present growth-fragmentation equations related to our model. In Section 3.2 we establish a convergence criterion of a sequence of OU type growth-fragmentations. In Section 3.3 we study the long-time asymptotic behavior.

3.1 Related growth-fragmentation equations

The evolution of the mean value of an OU type growth-fragmentation can be described by a growth-fragmentation equation.

Proposition 3.1. *Let $\mathbf{X} := (\mathbf{X}(t) = (X_1(t), X_2(t), \dots), t \geq 0)$ be an OU type growth-fragmentation process on c_0^\downarrow with characteristics (σ, c, ν, θ) , starting from $\mathbf{X}(0) = (1, 0, 0, \dots)$. For every $t \geq 0$, define a measure $\rho_{\mathbf{X}}(t)$ on $\mathbb{R}_+ = (0, \infty)$, such that for every $f \in C_c^\infty(\mathbb{R}_+)$ (the space of C^∞ -functions on \mathbb{R}_+ with compact support),*

$$\langle \rho_{\mathbf{X}}(t), f \rangle := \mathbb{E} \left[\sum_{i=1}^{\infty} f(X_i(t)) \right].$$

Then $(\rho_{\mathbf{X}}(t), t \geq 0)$ is a family of Radon measures and it solves the growth-fragmentation equation

$$\langle \rho_{\mathbf{X}}(t), f \rangle = f(1) + \int_0^t \langle \rho_{\mathbf{X}}(r), \mathcal{L}f \rangle dr, \quad (3.1)$$

where

$$\begin{aligned} \mathcal{L}f(x) := & \frac{1}{2}\sigma^2 x^2 f''(x) + \left(c + \frac{1}{2}\sigma^2 - \theta \log x\right) x f'(x) \\ & + \int_{\mathcal{S}} \left(\sum_{i=1}^{\infty} f(xs_i) - f(x) + x f'(x)(1 - s_1) \right) \nu(ds). \end{aligned} \quad (3.2)$$

See [15] and Corollary 3.12 in [11] for analogous results for self-similar growth-fragmentations; the proof of the latter relies on a remarkable genealogical martingale and a many-to-one theorem. However, we shall use a different approach, by first dealing with the truncated system and then passing to the limit.

To prove Proposition 3.1, we first show that $\mathcal{L}f$ is well-defined and continuous.

Lemma 3.2. *For every $f \in C_c^\infty(\mathbb{R}_+)$, $\mathcal{L}f$ is a continuous function on \mathbb{R}_+ and is identically zero in some neighborhood of zero. Furthermore, $\mathcal{L}f(x) = o(x^2)$ as $x \rightarrow \infty$.*

Proof. Set $\mathcal{L}_1 f(x) := \mathcal{L}f(x) + \theta \log(x) x f'(x)$, then we know from Lemma 2.1 in [15]¹ that $\mathcal{L}_1 f$ is continuous on \mathbb{R}_+ , identically zero in some neighborhood of zero and $\mathcal{L}_1 f(x) = o(x^2)$ as $x \rightarrow \infty$. It follows plainly that the same properties hold for $\mathcal{L}f$. \square

We next prove Proposition 3.1 for the finite branching case in the context of an OU type branching Markov chain.

Lemma 3.3. *Let \mathcal{Z} be an OU type branching Markov chain with characteristics (σ, c, μ, θ) , which satisfies $\mu(\mathcal{R} \setminus \mathcal{R}_1) < \infty$. For every $t \geq 0$, we may associated a Radon measure $\rho_{\mathcal{Z}}(t)$ such that for all $g \in C_c^\infty(\mathbb{R})$, the space of C^∞ -functions on \mathbb{R} with compact support, there is $\langle \rho_{\mathcal{Z}}(t), g \rangle := \mathbb{E} [\langle \mathcal{Z}(t), g \rangle]$. Then $(\rho_{\mathcal{Z}}(t), t \geq 0)$ is a solution of the equation*

$$\langle \rho_{\mathcal{Z}}(t), g \rangle = g(0) + \int_0^t \langle \rho_{\mathcal{Z}}(s), \mathcal{L}_{\mathcal{Z}} g \rangle ds,$$

where

$$\mathcal{L}_{\mathcal{Z}} g(z) = \frac{1}{2}\sigma^2 g''(z) + c g'(z) - \theta z g'(z) + \int_{\mathcal{R}} \left(\sum_{i=1}^{\infty} g(z + r_i) - g(z) + (1 - e^{r_1}) g'(z) \right) \mu(d\mathbf{r}).$$

¹though Lemma 2.1 in [15] is only concerned with the case when ν is binary and conservative, the same arguments work under our more general setting.

Proof. Recall the decomposition of \mathcal{Z} in Lemma 2.17 and the branching random walk description of $(\tilde{\mathcal{W}}(t) := e^{\theta t} \mathcal{W}(t), t \geq 0)$. By conditioning on $\mathcal{W}(t) := \{\{W_i, i \in I\}\}$ we have for every $g \in \mathcal{C}_c^\infty(\mathbb{R})$ that

$$\mathbb{E} [\langle \mathcal{Z}(t), g \rangle] = \mathbb{E} \left[\sum_{i \in I} g(e^{-\theta t} (e^{\theta t} W_i) + \beta_i) \right] = \mathbb{E} [\langle \tilde{\mathcal{W}}(t), Q_t g \rangle], \quad (3.3)$$

where $(Q_t)_{t \geq 0}$ denotes the semigroup of an OU type process with characteristics (ψ, θ) and we have used the scaling property (2.3) for the last equality. We know from [33] that the infinitesimal generator \mathcal{A} of $(Q_t)_{t \geq 0}$ has domain containing all C^2 functions on \mathbb{R} with compact support, and is given by

$$\begin{aligned} \mathcal{A}g(z) &:= \frac{1}{2} \sigma^2 g''(z) + \left(c + \int_{\mathcal{R} \setminus \mathcal{R}_1} (1 - e^{r_1}) \mu(d\mathbf{r}) \right) g'(z) - \theta z g'(z) \\ &\quad + \int_{\mathcal{R}_1} (g(z + r_1) - g(z) + (1 - e^{r_1}) g'(z)) \mu(d\mathbf{r}). \end{aligned}$$

It follows that $\frac{\partial}{\partial t} Q_t g = Q_t \mathcal{A}g = \mathcal{A}Q_t g$ for every $g \in \mathcal{C}_c^\infty(\mathbb{R})$. Using the classic stochastic analysis and the Poissonian construction of the branching random walk $\tilde{\mathcal{W}}$, we deduce for every $g \in \mathcal{C}_c^\infty(\mathbb{R})$ that

$$\begin{aligned} &\mathbb{E} [\langle \tilde{\mathcal{W}}(t), Q_t g \rangle] - g(0) \\ &= \mathbb{E} \left[\int_0^t \frac{\partial}{\partial s} \langle \tilde{\mathcal{W}}(s), Q_s g \rangle ds \right] + \mathbb{E} \left[\sum_{0 \leq s \leq t} (\langle \tilde{\mathcal{W}}(s), Q_s g \rangle - \langle \tilde{\mathcal{W}}(s-), Q_s g \rangle) \right] \\ &= \mathbb{E} \left[\int_0^t \langle \tilde{\mathcal{W}}(s), Q_s \mathcal{A}g \rangle ds \right] + \int_0^t ds \int_{\mathcal{R} \setminus \mathcal{R}_1} \sum_{k=1}^{\infty} \mathbb{E} \left[\sum_{w \in \tilde{\mathcal{W}}(s-)} (Q_s g(w + e^{\theta s} r_k) - Q_s g(w)) \right] \mu(d\mathbf{r}). \end{aligned}$$

Using again (3.3), we have

$$\begin{aligned} &\mathbb{E} [\langle \mathcal{Z}(t), g \rangle] - g(0) \\ &= \int_0^t \mathbb{E} [\langle \mathcal{Z}(s), \mathcal{A}g \rangle] ds + \int_0^t ds \int_{\mathcal{R} \setminus \mathcal{R}_1} \sum_{k=1}^{\infty} \mathbb{E} \left[\sum_{z \in \mathcal{Z}(s-)} (g(z + r_k) - g(z)) \right] \mu(d\mathbf{r}). \end{aligned}$$

By the definition of $\mathcal{L}_{\mathcal{Z}}$, this is indeed the desired result. \square

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. By the linearity, it suffices to prove for the case when $f \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ is further non-negative. Fix $t \geq 0$ and $q \geq 2(e^{\theta t} \vee 1)$. We deduce from Lemma 3.2 that there exists a constant $C > 0$ such that $|\mathcal{L}f(x)| \leq C|x|^q$ for every $x \in \mathbb{R}_+$. Then Theorem 2.8 leads to

$$\mathbb{E} \left[\sum_{i=1}^{\infty} |\mathcal{L}f(X_i(r))| \right] \leq C \mathbb{E} \left[\sum_{i=1}^{\infty} X_i(r)^q \right] = C \exp \left(\int_0^r \kappa(qe^{-\theta s}) ds \right), \quad \text{for all } r \in [0, t].$$

This entails that

$$\int_0^t \mathbb{E} \left[\sum_{i=1}^{\infty} |\mathcal{L}f(X_i(r))| \right] dr < \infty. \quad (3.4)$$

By Theorem 2.8 we also have $\mathbb{E} [\sum_{i=1}^{\infty} f(X_i(t))] < \infty$.

Let us next consider for every $\ell > 0$ the truncated OU type growth-fragmentation $\mathbf{X}^{(\ell)}$ and its associated OU type branching Markov chain $\mathcal{Z}^{(\ell)}$, with relation

$$\langle \mathbf{X}^{(\ell)}(t), f \rangle := \sum_{i=1}^{\infty} f(X_i^{(\ell)}(t)) = \langle \mathcal{Z}^{(\ell)}(t), f \circ \exp \rangle.$$

Applying Lemma 3.3 to $\mathcal{Z}^{(\ell)}$, we deduce that

$$\mathbb{E} \left[\langle \mathbf{X}^{(\ell)}(t), f \rangle \right] = f(1) + \mathbb{E} \left[\int_0^t \langle \mathbf{X}^{(\ell)}(s), \mathcal{L}^{(\ell)} f \rangle ds \right],$$

where

$$\begin{aligned} \mathcal{L}^{(\ell)} f(y) &:= \frac{1}{2} \sigma^2 y^2 f''(y) + \left(c + \frac{1}{2} \sigma^2 - \theta \log y \right) y f'(y) \\ &+ \int_{\mathcal{S}} \left(f(y s_1) - f(y) + \sum_{i \geq 2} f(y s_i) \mathbb{1}_{\{s_i \geq e^{-\ell}\}} + y f'(y) (1 - s_1) \right) \nu(ds). \end{aligned}$$

Letting $\ell \uparrow \infty$, we immediately check by monotone convergence that (since f is non-negative)

$$\lim_{\ell \uparrow \infty} \mathcal{L}^{(\ell)} f(y) = \mathcal{L} f(y), \quad y \geq 0.$$

Recall from (2.16) that $\lim_{t \rightarrow \infty} \mathbf{X}^{(\ell)}(r) = \mathbf{X}(r)$ for all $r \geq 0$, we hence obtain by dominated convergence (ensured by (3.4)) that

$$\lim_{\ell \uparrow \infty} \int_0^t \mathbb{E} \left[\langle \mathbf{X}^{(\ell)}(r), \mathcal{L}^{(\ell)} f \rangle \right] dr = \int_0^t \mathbb{E} [\langle \mathbf{X}(r), \mathcal{L} f \rangle] dr.$$

On the other hand, we deduce by monotone convergence that

$$\lim_{\ell \uparrow \infty} \mathbb{E} \left[\langle \mathbf{X}^{(\ell)}(t), f \rangle \right] = \mathbb{E} [\langle \mathbf{X}(t), f \rangle].$$

So we conclude that

$$\mathbb{E} [\langle \mathbf{X}(t), f \rangle] = f(1) + \int_0^t \mathbb{E} [\langle \mathbf{X}(r), \mathcal{L} f \rangle] dr,$$

which means that $\rho_{\mathbf{X}}$ is indeed a Radon measure on \mathbb{R}_+ and is a solution of (3.1). \square

3.2 Convergence of OU type growth-fragmentations

For every $n \in \bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, let \mathbf{X}_n be an OU type growth-fragmentation with characteristics $(\sigma_n, c_n, \nu_n, \theta_n)$ starting from $\mathbf{1} := (1, 0, \dots)$ and κ_n be its cumulant. In this subsection we establish the following convergence result.

Theorem 3.4. *Suppose that*

$$\nu_n((0, 0, \dots)) = 0 \text{ for all } n \in \bar{\mathbb{N}}, \quad (3.5)$$

that

$$\lim_{n \rightarrow \infty} \theta_n = \theta_{\infty}, \quad (3.6)$$

that

$$\lim_{n \rightarrow \infty} (c_n + \sigma_n^2/2) = c_{\infty} + \sigma_{\infty}^2/2, \quad (3.7)$$

and that there is the weak convergence of finite measures on \mathcal{S}

$$\sigma_n^2 \delta_1(ds) + (1 - s_1)^2 \nu_n(ds) \xrightarrow{n \rightarrow \infty} \sigma_\infty^2 \delta_1(ds) + (1 - s_1)^2 \nu_\infty(ds). \quad (3.8)$$

Write $\bar{\theta} := \sup_{n \in \bar{\mathbb{N}}} \theta_n < \infty$, then for every $T \geq 0$ and $q > 2(e^{\bar{\theta}T} \vee 1)$, there is the weak convergence

$$\left(\mathbf{X}_n(t), t \in [0, T] \right) \xrightarrow{n \rightarrow \infty} \left(\mathbf{X}_\infty(t), t \in [0, T] \right),$$

in the space $D([0, T], \ell^{q\downarrow})$ of càdlàg functions with values in $\ell^{q\downarrow}$ endowed with the Skorokhod topology. As a consequence, the weak convergence

$$\mathbf{X}_n \xrightarrow{n \rightarrow \infty} \mathbf{X}_\infty,$$

holds in the space $D(\mathbb{R}_+, c_o^\downarrow)$ of càdlàg functions with values in c_o^\downarrow endowed with the Skorokhod topology.

This result generalizes Theorem 2 in [9], which deals with the case $\theta_n \equiv 0$ for every $n \in \bar{\mathbb{N}}$; the assumptions (3.5), (3.7) and (3.8) are inherited from there.² The condition (3.5) is a minor technical assumption that makes our arguments less cumbersome.

Remark 3.5. The reason for which we consider the space $\ell^{q\downarrow}$ with $q > 2(e^{\bar{\theta}T} \vee 1)$ is as follows. Recall that $[2, \infty) \subset \text{dom}(\kappa_n)$ for all $n \in \bar{\mathbb{N}}$, then $\mathbf{X}_n(t) \in \ell^{2(e^{\bar{\theta}T} \vee 1)\downarrow}$ (by Theorem 2.8) for every $t \in [0, T]$. We further need to enlarge the state space to $\ell^{q\downarrow}$ with $q > 2(e^{\bar{\theta}T} \vee 1)$, so as to ensure that $(\mathbf{X}_n(t))_{n \in \bar{\mathbb{N}}}$ is tight in $\ell^{q\downarrow}$, which does not necessarily hold with $q = 2(e^{\bar{\theta}T} \vee 1)$. See the proof of Lemma 3.10 below for details.

Before tackling the proof of Theorem 3.4, we point out several evidences that suggest its validity. Firstly, (3.7) and (3.8) yield the convergence of the cumulant

$$\lim_{n \rightarrow \infty} \kappa_n(p) = \kappa_\infty(p), \quad \text{for all } p > 2. \quad (3.9)$$

However, this convergence does not necessarily hold for $p = 2$. Secondly, we have the convergence of the selected fragments defined as in Lemma 2.11. Indeed, one easily deduces from (3.7) and (3.8) the convergence of the Laplace exponents (2.13):

$$\lim_{n \rightarrow \infty} \Phi_{n,*}(p) = \Phi_{\infty,*}(p), \quad \text{for all } p \geq 0. \quad (3.10)$$

Then the convergence of the selected fragments is a consequence of the following lemma.

Lemma 3.6. For every $n \in \bar{\mathbb{N}}$, let Z_n be an OU type process with characteristics $(\Phi_{n,*}, \theta_n)$ starting from $Z_n(0) = 0$. Suppose that (3.6) and (3.10) hold. Then there exists a coupling of $(Z_n, n \in \bar{\mathbb{N}})$, such that for every $t \geq 0$

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, t]} |Z_n(s) - Z_\infty(s)| = 0, \quad \text{in probability.}$$

Proof. Recall from (2.3) that Z_n is a stochastic integral

$$Z_n(t) = \int_0^t e^{-\theta_n(t-s)} d\xi_n(s), \quad t \geq 0,$$

where ξ_n is a Lévy process with Laplace exponent $\Phi_{n,*}$. We first observe that there exists a coupling of Lévy processes $(\xi_n)_{n \in \bar{\mathbb{N}}}$, such that for every $t \geq 0$

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, t]} |\xi_n(s) - \xi_\infty(s)| = 0, \quad \text{in probability;}$$

²There is a typo in Theorem 2 in [9] for the condition (3.7).

see e.g. Theorem 15.14 and 15.17 in [26]. Therefore, an application of Theorem 5 in [24] leads to the claim, if $(\xi_n)_{n \in \mathbb{N}}$ satisfy the so-called *condition UT*. To check the *condition UT*, we shall use Lemme 3.1 in [25]. Consider $\xi_n^1(t) := \xi_n(t) - \sum_{|\Delta \xi_n(s)| > 1} \Delta \xi_n(s)$. Then $b_n^1 := \mathbb{E} [\xi_n^1(1)]$ is finite, and $M_n^1(t) := \xi_n^1(t) - b_n^1 t$ is a martingale. In other words, the canonical decomposition of the special semimartingale ξ_n^1 is given by

$$\xi_n^1(t) = b_n^1 t + M_n^1(t).$$

The family of the variations of the processes $(b_n^1 t)_{t \geq 0}$ is clearly tight, then it follows from Lemme 3.1 in [25] that (ξ_n) satisfy the *condition UT*. \square

The rest of this subsection is devoted to the proof of Theorem 3.4. By Prokhorov's theorem (see e.g. Section 5 in [18]), we shall prove the weak convergence of finite dimensional distributions and the tightness. In the remaining of this subsection, we fix $\bar{\theta} := \sup_{n \in \mathbb{N}} \theta_n < \infty$, $T \geq 0$ and $q > 2(e^{\bar{\theta}T} \vee 1)$.

Convergence of finite dimensional distributions The proof of the weak convergence of finite dimensional distributions proceeds as Lemma 7 in [9]. Consider for every $n \in \mathbb{N}$ and $\ell \geq 0$ the truncated OU type growth-fragmentation $\mathbf{X}_n^{(\ell)}$. Recall that $\mathbf{X}_n^{(\ell)}$ corresponds to an OU type branching Markov chain $\mathcal{Z}_n^{(\ell)}$ with characteristics $(\sigma_n, c_n, \mu_n^{(\ell)}, \theta_n)$, where $\mu_n^{(\ell)}$ is the image of ν by the map $(s_1, s_2, \dots) \mapsto (\log s_1, \log s_2, \dots)^{(\ell)}$ as in (2.8).

Lemma 3.7 (Lemma 6 in [9]). *Suppose that (3.5) and (3.8) hold. Then for every $\ell \geq 0$, there is the weak convergence of finite measures on \mathcal{R}*

$$\sigma_n^2 \delta_{(0, -\infty, \dots)}(d\mathbf{r}) + (1 - e^{r_1})^2 \mu_n^{(\ell)}(d\mathbf{r}) \xrightarrow{n \rightarrow \infty} \sigma_\infty^2 \delta_{(0, -\infty, \dots)}(d\mathbf{r}) + (1 - e^{r_1})^2 \mu_\infty^{(\ell)}(d\mathbf{r}),$$

and

$$\mu_n^{(\ell)}(\cdot | \mathcal{R} \setminus \mathcal{R}_1) \xrightarrow{n \rightarrow \infty} \mu_\infty^{(\ell)}(\cdot | \mathcal{R} \setminus \mathcal{R}_1).$$

These relations lead to the following convergence.

Lemma 3.8. *Suppose that (3.5), (3.6), (3.7) and (3.8) hold. Then for every $\ell \geq 0$, there exists a coupling of $(\mathbf{X}_n^{(\ell)})_{n \in \mathbb{N}}$, such that for every $t \geq 0$ and $p \geq 2$,*

$$\lim_{n \rightarrow \infty} \|\mathbf{X}_n^{(\ell)}(t) - \mathbf{X}_\infty^{(\ell)}(t)\|_{\ell^p} = 0 \quad \text{in probability.}$$

Proof. Recall that in the construction of $\mathcal{Z}_n^{(\ell)}$ by Definition 2.4, each particle $u \in \mathbb{U}$ is born at time $b_{n,u} \geq 0$ with initial position $a_{n,u}$, and then moves according to an OU type process $Z_{n,u}^{(\ell)}$ with characteristics $(\psi_n^{(\ell)}, \theta_n)$, where $\psi_n^{(\ell)}$ is given by (2.7). After an exponential time $\lambda_{n,u}^{(\ell)}$ with parameter $\nu_n^{(\ell)}(\mathcal{S} \setminus \mathcal{S}_1)$, it splits into at most $\lceil e^\ell \rceil$ particles whose relative positions are $(\Delta a_{n,ui}^{(\ell)}, i \in \mathbb{N})$, distributed according to $\nu_n^{(\ell)}(\cdot | \mathcal{S} \setminus \mathcal{S}_1)$. We shall prove that there exists a coupling of $(\mathcal{Z}_n^{(\ell)})_{n \in \mathbb{N}}$, such that the following sequences indexed by \mathbb{U}

$$\left(\mathbb{1}_{\{b_{n,u}^{(\ell)} \leq t < b_{n,u}^{(\ell)} + \lambda_{n,u}^{(\ell)}\}} \exp(e^{-\theta_n(t - b_{n,u}^{(\ell)})} a_{n,u}^{(\ell)} + Z_{n,u}^{(\ell)}(t - b_{n,u}^{(\ell)})), \quad u \in \mathbb{U} \right)$$

converges in probability as $n \rightarrow \infty$, for ℓ^p -distance. Then the claim follows since the rearrangement of sequences in decreasing order decreases the ℓ^p -distance.

For every $u \in \mathbb{U}$, we may assume by Lemma 3.7 and Skorokhod representation theorem that the random variables $\lambda_{n,u}^{(\ell)}$, $\Delta a_{n,u}^{(\ell)}$ are coupled in such a way that

$$\lim_{n \rightarrow \infty} \lambda_{n,u}^{(\ell)} = \lambda_{\infty,u}^{(\ell)}, \quad \text{a.s.} \tag{3.11}$$

and

$$\lim_{n \rightarrow \infty} \Delta a_{n,ui}^{(\ell)} = \Delta a_{\infty,ui}^{(\ell)}, \quad \text{for all } i \in \mathbb{N}, \text{ a.s.}$$

We further deduce from (3.7) and Lemma 3.7 that $\lim_{n \rightarrow \infty} \psi_n^{(\ell)}(p) = \psi_\infty^{(\ell)}(p)$ for every $p \geq 0$. Using Lemma 3.6 leads to

$$\lim_{n \rightarrow \infty} Z_{n,u}^{(\ell)}(s) = Z_{\infty,u}^{(\ell)}(s), \quad \text{for all } s > 0, \text{ a.s.}$$

Therefore, for every $u \in \mathbb{U}$, we have

$$\lim_{n \rightarrow \infty} \exp(-\theta_n(t - b_{n,u}^{(\ell)})) a_{n,u}^{(\ell)} + Z_{n,u}^{(\ell)}(t - b_{n,u}^{(\ell)}) = \exp(-\theta_\infty(t - b_{\infty,u}^{(\ell)})) a_{\infty,u}^{(\ell)} + Z_{\infty,u}^{(\ell)}(t - b_{\infty,u}^{(\ell)}), \quad \text{a.s.}$$

Denote the set of vertices alive at time $t \geq 0$ by $V_{n,t} \subset \mathbb{U}$. Observe that $V_{n,t}$ is almost surely a finite set; further, it follows from (3.11) that $V_{n,t}$ coincides with $V_{\infty,t}$ with high probability. Summarizing, we have completed the proof. \square

We also need the following estimation.

Lemma 3.9. *For every $t \geq 0$ and $p \geq 2(e^{\bar{\theta}t} \vee 1)$, there is*

$$\lim_{\ell \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E} \left[\|\mathbf{X}_n(t) - \mathbf{X}_n^{(\ell)}(t)\|_{\ell^p}^p \right] = 0$$

Proof. We deduce from (2.17) and Theorem 2.8 that

$$\mathbb{E} \left[\|\mathbf{X}_n(t) - \mathbf{X}_n^{(\ell)}(t)\|_{\ell^p}^p \right] \leq K_n(p, t) - K_n^{(\ell)}(p, t) = K_n(p, t) \left(1 - \exp \left(\int_0^t -(\kappa_n(pe^{-\theta_n r}) - \kappa_n^{(\ell)}(pe^{-\theta_n r})) dr \right) \right),$$

where $K_n(p, t) := \exp \left(\int_0^t \kappa_n(pe^{-\theta_n r}) dr \right)$ and $K_n^{(\ell)}(p, t) := \exp \left(\int_0^t \kappa_n^{(\ell)}(pe^{-\theta_n r}) dr \right)$. Since for every $\mathbf{s} = (s_1, s_2, \dots) \in \mathcal{S}$, there is

$$\sum_{i=2}^{\infty} \mathbb{1}_{\{s_i \leq e^{-\ell}\}} s_i^{pe^{-\theta_n r}} \leq e^{-\ell(pe^{-\theta_n r} - 2)} \sum_{i=2}^{\infty} s_i^2 \leq e^{-\ell(pe^{-\theta_n r} - 2)} (1 - s_1)^2,$$

we have

$$\kappa_n(pe^{-\theta_n r}) - \kappa_n^{(\ell)}(pe^{-\theta_n r}) = \int_{\mathcal{S}} \sum_{i=2}^{\infty} \mathbb{1}_{\{s_i \leq e^{-\ell}\}} s_i^{pe^{-\theta_n r}} \nu_n(d\mathbf{s}) \leq e^{-\ell(pe^{-\theta_n r} - 2)} \int_{\mathcal{S}} (1 - s_1)^2 \nu_n(d\mathbf{s}).$$

It follows that

$$\mathbb{E} \left[\|\mathbf{X}_n(t) - \mathbf{X}_n^{(\ell)}(t)\|_{\ell^p}^p \right] \leq K_n(p, t) \left(1 - \exp \left(- \int_{\mathcal{S}} (1 - s_1)^2 \nu_n(d\mathbf{s}) \int_0^t e^{-\ell(pe^{-\theta_n r} - 2)} dr \right) \right).$$

As $p > 2(e^{\bar{\theta}t} \vee 1)$, we have $\inf_{n \in \mathbb{N}, r \in [0, t]} (pe^{-\theta_n r} - 2) > 0$. We also deduce from (3.8) and (3.9) that

$$\sup_{n \in \mathbb{N}} \int_{\mathcal{S}} (1 - s_1)^2 \nu_n(d\mathbf{s}) < \infty, \tag{3.12}$$

and that

$$\sup_{n \in \mathbb{N}} K_n(p, t) \leq \sup_{n \in \mathbb{N}} \exp \left(\int_0^t |\kappa_n(pe^{-\theta_n s})| ds \right) < \infty. \tag{3.13}$$

Then the claim follows. \square

We are now ready to prove the weak convergence of finite-dimensional distributions.

Lemma 3.10. *Suppose that (3.5), (3.6), (3.7) and (3.8) hold, then Theorem 3.4 holds for finite-dimensional distributions in $\ell^{q\downarrow}$.*

Proof. For simplicity, we shall only establish the convergence for one-dimensional; similar arguments hold for multi-dimensional case.

We first claim that for $q' \in (2(e^{\bar{\theta}T} \vee 1), q)$, the set

$$B_r := \{\mathbf{x} \in \ell^{q\downarrow} : \|\mathbf{x}\|_{\ell^{q'}} \leq r\},$$

is a compact subset in $\ell^{q\downarrow}$. Indeed, for any sequence in B_r , we may use the diagonal procedure to extract a subsequence that converges pointwisely, and the limit belongs to B_r due to Fatou's lemma. Since B_r is equi-summable in ℓ^q (because $q' < q$), the convergence also holds for ℓ^q -distance. Next, it follows from Theorem 2.8 that

$$\mathbb{P}(\mathbf{X}_n(t) \notin B_r) \leq r^{-q'} \mathbb{E} \left[\|\mathbf{X}_n(t)\|_{\ell^{q'}}^{q'} \right] = r^{-q'} \exp \left(\int_0^t \kappa_n(q' e^{-\theta_n r}) dr \right), \quad t \in [0, T].$$

We hence deduce from (3.13) that the sequence $(\mathbf{X}_n(t), n \in \bar{\mathbb{N}})$ is tight in $\ell^{q\downarrow}$.

So it remains to prove the uniqueness of the limit of a converging subsequence. Let $k \in \mathbb{N}$ and $F : \mathbb{R}_+^k \rightarrow [0, 1]$ be a continuous function. For every $\mathbf{x} = (x_1, x_2, \dots) \in \ell^{q\downarrow}$, write $F(\mathbf{x}) := F(x_1, \dots, x_k)$. Then F is continuous on $\ell^{q\downarrow}$. We shall prove for every $t \in [0, T]$ that

$$\lim_{n \rightarrow \infty} \mathbb{E}[F(\mathbf{X}_n(t))] = \mathbb{E}[F(\mathbf{X}_\infty(t))].$$

If this holds for every $k \in \mathbb{N}$ and such function F , then we deduce the uniqueness of the limit.

For every $\ell \geq 0$ there is

$$\begin{aligned} & \left| \mathbb{E}[F(\mathbf{X}_n(t)) - F(\mathbf{X}_\infty(t))] \right| \\ & \leq \left| \mathbb{E}[F(\mathbf{X}_n^{(\ell)}(t)) - F(\mathbf{X}_\infty^{(\ell)}(t))] \right| + \left| \mathbb{E}[F(\mathbf{X}_n(t)) - F(\mathbf{X}_n^{(\ell)}(t))] \right| + \left| \mathbb{E}[F(\mathbf{X}_\infty(t)) - F(\mathbf{X}_\infty^{(\ell)}(t))] \right|. \end{aligned}$$

Let us estimate these three terms. Fix an arbitrarily small $\epsilon > 0$. By the tightness of $(\mathbf{X}_n(t), n \in \bar{\mathbb{N}})$ we may choose $r > 0$ large enough such that

$$\mathbb{P}(\mathbf{X}_n(t) \notin B_r) < \epsilon \quad \text{for every } n \in \bar{\mathbb{N}}.$$

Note that if $\mathbf{X}_n(t) \in B_r$, then $\mathbf{X}_n^{(\ell)}(t) \in B_r$ for every $\ell \geq 0$. So we have

$$\mathbb{P}(\mathbf{X}_n^{(\ell)}(t) \notin B_r) \leq \mathbb{P}(\mathbf{X}_n(t) \notin B_r) < \epsilon.$$

As F is uniformly continuous on the compact subset B_r in $\ell^{q\downarrow}$, there exists $\eta > 0$ such that

$$|F(\mathbf{x}) - F(\mathbf{x}')| < \epsilon, \quad \text{for all } \mathbf{x}, \mathbf{x}' \in B_r \text{ with } \|\mathbf{x} - \mathbf{x}'\|_{\ell^q} < \eta.$$

Using Lemma 3.9 and Markov inequality, we next choose ℓ large enough such that

$$\sup_{n \in \bar{\mathbb{N}}} \mathbb{P}(\|\mathbf{X}_n(t) - \mathbf{X}_n^{(\ell)}(t)\|_{\ell^q} \geq \eta) \leq \epsilon.$$

We hence deduce that

$$\begin{aligned} & \left| \mathbb{E} \left[F(\mathbf{X}_n(t)) - F(\mathbf{X}_n^{(\ell)}(t)) \right] \right| \\ & \leq \mathbb{P} \left(\mathbf{X}_n^{(\ell)}(t) \notin B_r \right) + \mathbb{P} \left(\mathbf{X}_n(t) \notin B_r \right) + \mathbb{P} \left(\|\mathbf{X}_n(t) - \mathbf{X}_n^{(\ell)}(t)\|_{\ell^q} \geq \eta \right) + \epsilon < 4\epsilon, \quad \text{for all } n \in \bar{\mathbb{N}}. \end{aligned}$$

It remains to estimate $\left| \mathbb{E} \left[F(\mathbf{X}_n^{(\ell)}(t)) - F(\mathbf{X}_\infty^{(\ell)}(t)) \right] \right|$. By Lemma 3.8, we may choose n large enough such that $\mathbb{P} \left(\|\mathbf{X}_n^{(\ell)}(t) - \mathbf{X}_\infty^{(\ell)}(t)\|_{\ell^q} \geq \eta \right) < \epsilon$. It follows that

$$\begin{aligned} & \left| \mathbb{E} \left[F(\mathbf{X}_n^{(\ell)}(t)) - F(\mathbf{X}_\infty^{(\ell)}(t)) \right] \right| \\ & \leq \mathbb{P} \left(\mathbf{X}_n^{(\ell)}(t) \notin B_r \right) + \mathbb{P} \left(\mathbf{X}_\infty^{(\ell)}(t) \notin B_r \right) + \mathbb{P} \left(\|\mathbf{X}_n^{(\ell)}(t) - \mathbf{X}_\infty^{(\ell)}(t)\|_{\ell^q} \geq \eta \right) + \epsilon < 4\epsilon. \end{aligned}$$

We have completed the proof. \square

Tightness We finally complete the proof of Theorem 3.4 by checking Aldous' tightness criterion (see e.g. Theorem 16.11 in [26]).

Lemma 3.11. *Let $(h_n, n \in \mathbb{N})$ be a sequence of constants with $h_n > 0$ and $\lim_{n \rightarrow \infty} h_n = 0$, and $(\tau_n, n \in \mathbb{N})$ be a sequence of \mathbf{X}_n -stopping times with $\tau_n < T$ almost surely. Suppose that (3.5), (3.6), (3.7) and (3.8) hold, then we have for every $q > 2(e^{\bar{\theta}T} \vee 1)$ with $\bar{\theta} := \sup_{n \in \bar{\mathbb{N}}} \theta_n$,*

$$\lim_{n \rightarrow \infty} \|\mathbf{X}_n(\tau_n) - \mathbf{X}_n(\tau_n + h_n)\|_{\ell^q} = 0 \quad \text{in probability.}$$

Proof. Denote $\mathbf{X}_n(\tau_n) := (X_{n,1}(\tau_n), X_{n,2}(\tau_n), \dots)$ and

$$\mathbf{X}_n(\tau_n)^{e^{-\theta_n h_n}} := (X_{n,1}(\tau_n)^{e^{-\theta_n h_n}}, X_{n,2}(\tau_n)^{e^{-\theta_n h_n}}, \dots).$$

An elementary inequality leads to

$$\begin{aligned} & \mathbb{E} \left[\|\mathbf{X}_n(\tau_n) - \mathbf{X}_n(\tau_n + h_n)\|_{\ell^q}^q \right] \\ & \leq 2^{q-1} \left(\mathbb{E} \left[\|\mathbf{X}_n(\tau_n) - \mathbf{X}_n(\tau_n)^{e^{-\theta_n h_n}}\|_{\ell^q}^q \right] + \mathbb{E} \left[\|\mathbf{X}_n(\tau_n)^{e^{-\theta_n h_n}} - \mathbf{X}_n(\tau_n + h_n)\|_{\ell^q}^q \right] \right). \end{aligned} \quad (3.14)$$

We shall evaluate the two expected values in (3.14) respectively. Let us start with the first one. Applying the mean value theorem to the function $x \mapsto X_{n,i}(\tau_n)^x$, we obtain that

$$|X_{n,i}(\tau_n) - X_{n,i}(\tau_n)^{e^{-\theta_n h_n}}| \leq \mathbb{1}_{\{X_{n,i}(\tau_n) > 0\}} \max \left(X_{n,i}(\tau_n), X_{n,i}(\tau_n)^{e^{-\theta_n h_n}} \right) \left| \log(X_{n,i}(\tau_n)) \right| |1 - e^{-\theta_n h_n}|.$$

Denote $c_I := \inf_{n \in \mathbb{N}} e^{-\theta_n h_n}$ and $c_S := \sup_{n \in \mathbb{N}} e^{-\theta_n h_n}$, then

$$\max \left(X_{n,i}(\tau_n), X_{n,i}(\tau_n)^{e^{-\theta_n h_n}} \right) \leq X_{n,i}(\tau_n) + X_{n,i}(\tau_n)^{c_I} + X_{n,i}(\tau_n)^{c_S}.$$

As $h_n \rightarrow 0$ and $q > 2(e^{\bar{\theta}T} \vee 1)$, without loss of generality, we may assume that $\sup_{n \in \mathbb{N}} |h_n|$ is small enough such that $q c_I > 2(e^{\bar{\theta}T} \vee 1)$. Then fix $\delta > 0$ such that $q(c_I \wedge 1)(1 - \delta) > 2(e^{\bar{\theta}T} \vee 1)$. It is elementary to see that there exists $c_\delta > 0$ such that $|\log x| \leq c_\delta(x^\delta + x^{-\delta})$ for all $x > 0$, then we have

$$\|\mathbf{X}_n(\tau_n) - \mathbf{X}_n(\tau_n)^{e^{-\theta_n h_n}}\|_{\ell^q}^q \leq c_\delta |1 - e^{-\theta_n h_n}| \sum_{k=1}^6 \left(\sum_{i=1}^{\infty} X_{n,i}(\tau_n)^{q_k} \right),$$

where $\{q_k\}$ are constants $\{qc_I \pm q\delta, q \pm q\delta, qc_S \pm q\delta\}$. These constants are all greater than $2(e^{\bar{\theta}T} \vee 1)$ thanks to the choice of δ , so we can obtain martingales by Proposition 2.13. As $\tau_n < T$ a.s., using the optional stopping theorem to these martingales, we have

$$\mathbb{E} \left[\left\| \mathbf{X}_n(\tau_n) - \mathbf{X}_n(\tau_n) e^{-\theta_n h_n} \right\|_{\ell^q}^q \right] \leq c_\delta |1 - e^{-\theta_n h_n}| \sum_{k=1}^6 \exp \left(\int_0^T |\kappa_n(q_k e^{-\theta_n r})| dr \right).$$

As $\theta_n h_n \rightarrow 0$ and (3.13) holds, we hence deduce that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left\| \mathbf{X}_n(\tau_n) - \mathbf{X}_n(\tau_n) e^{-\theta_n h_n} \right\|_{\ell^q}^q \right] = 0.$$

We next proceed to the second term in (3.14). From the strong Markov property (see Proposition 2.12 and Remark 2.16), we have that

$$\mathbb{E} \left[\left\| \mathbf{X}_n(\tau_n) e^{-\theta_n h_n} - \mathbf{X}_n(\tau_n + h_n) \right\|_{\ell^q}^q \right] = \mathbb{E} \left[\left\| \mathbf{X}_n(\tau_n) e^{-\theta_n h_n} \right\|_{\ell^q}^q \right] \mathbb{E} \left[\left\| \mathbf{X}_n(h_n) - (1, 0, \dots) \right\|_{\ell^q}^q \right].$$

Again, as $\tau_n \leq T$ a.s., it follows from Proposition 2.13 and the optional stopping theorem that

$$\mathbb{E} \left[\left\| \mathbf{X}_n(\tau_n) e^{-\theta_n h_n} \right\|_{\ell^q}^q \right] \leq \exp \left(\int_0^T |\kappa_n(q e^{-\theta_n h_n} e^{-\theta_n s})| ds \right) \leq \exp \left(\int_0^{T+h_n} |\kappa_n(q e^{-\theta_n s})| ds \right).$$

We hence deduce from (3.13) that

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[\left\| \mathbf{X}_n(\tau_n) e^{-\theta_n h_n} \right\|_{\ell^q}^q \right] \leq \sup_{n \in \mathbb{N}} \exp \left(\int_0^{T+h_n} |\kappa_n(q e^{-\theta_n s})| ds \right) < \infty. \quad (3.15)$$

Write $\tilde{\mathbf{X}}_n(h_n)$ for the sequence obtained from $\mathbf{X}_n(h_n)$ by exchanging the selected fragment $X_{n,*}(h_n)$ (see Lemma 2.11) and the largest one. Rearranging sequences in decreasing order reduces the ℓ^q -distance, so

$$\mathbb{E} \left[\left\| \mathbf{X}_n(h_n) - (1, 0, \dots) \right\|_{\ell^q}^q \right] \leq \mathbb{E} \left[\left\| \tilde{\mathbf{X}}_n(h_n) - (1, 0, \dots) \right\|_{\ell^q}^q \right].$$

Further, it follows from Lemma 2.11 and Theorem 2.8 that

$$\mathbb{E} \left[\left\| \tilde{\mathbf{X}}_n(h_n) - (1, 0, \dots) \right\|_{\ell^q}^q \right] = \mathbb{E} [|X_{n,*}(h_n) - 1|^q] + \exp \left(\int_0^{h_n} \kappa_n(q e^{-\theta_n s}) ds \right) - \exp \left(\int_0^{h_n} \Phi_{n,*}(q e^{-\theta_n s}) ds \right).$$

On the one hand, take an even integer $N > q$, then by Hölder's inequality:

$$\begin{aligned} \mathbb{E} [|X_{n,*}(h_n) - 1|^q] &\leq \mathbb{E} [|X_{n,*}(h_n) - 1|^N]^{q/N} \\ &= \mathbb{E} \left[\sum_{k=0}^N \binom{N}{k} (-1)^{N-k} X_{n,*}(h_n)^k \right]^{q/N} \\ &= \left(\sum_{k=0}^N \binom{N}{k} (-1)^{N-k} \exp \left(\int_0^{h_n} \Phi_{n,*}(k e^{-\theta_n s}) ds \right) \right)^{q/N}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \Phi_{n,*}(p) = \Phi_{\infty,*}(p)$ for every $p \geq 0$, we deduce that

$$\lim_{n \rightarrow \infty} \exp \left(\int_0^{h_n} \Phi_{n,*}(k e^{-\theta_n s}) ds \right) = 1, \quad \text{for every } k = 0, 1, \dots, N,$$

which leads to

$$\lim_{n \rightarrow \infty} \mathbb{E} [|X_{n,*}(h_n) - 1|^q] = 0.$$

On the other hand, for every $p \geq 2$, there is

$$\kappa_n(p) - \Phi_{n,*}(p) = \int_{\mathcal{S}} \sum_{i=2}^{\infty} s_i^p d\mathbf{s} \leq \int_{\mathcal{S}} (1 - s_1)^2 \nu_n(d\mathbf{s}).$$

Then we have

$$\begin{aligned} & \exp \left(\int_0^{h_n} \kappa_n(qe^{-\theta_n s}) ds \right) - \exp \left(\int_0^{h_n} \Phi_{n,*}(qe^{-\theta_n s}) ds \right) \\ & \leq \exp \left(\int_0^{h_n} \kappa_n(qe^{-\theta_n s}) ds \right) \left(1 - \exp \left(-h_n \int_{\mathcal{S}} (1 - s_1)^2 \nu_n(d\mathbf{s}) \right) \right). \end{aligned} \quad (3.16)$$

Since (3.12) and (3.13) hold, then (3.16) converges to 0 as $n \rightarrow \infty$. We hence conclude that

$$\lim_{n \rightarrow \infty} \mathbb{E} [\|\mathbf{X}_n(h_n) - (1, 0, \dots)\|_{\ell^q}^q] = 0.$$

This and (3.15) entail that

$$\lim_{n \rightarrow \infty} \mathbb{E} [\|\mathbf{X}_n(\tau_n)e^{-\theta_n h_n} - \mathbf{X}_n(\tau_n + h_n)\|_{\ell^q}^q] = 0.$$

We have completed the proof. \square

3.3 A law of large numbers for the inward case

In this subsection we fix an OU type growth-fragmentation \mathbf{X} with characteristics (σ, c, ν, θ) and cumulant κ , and always suppose that \mathbf{X} is **inward**, i.e. $\theta > 0$. We shall study the long-time asymptotic behavior of \mathbf{X} . Roughly speaking, our main result, Corollary 3.16, shows that the average of the sizes of the fragments converges to a stationary distribution as time tends to infinity.

Before stating our results, let us discuss the required assumptions. To make sense of the “average size”, we naturally need that the number of fragments is finite and non-zero at all time. In this direction, we suppose that the cumulant κ satisfies

$$\kappa(0) = \int_{\mathcal{S}} (\#\mathbf{s} - 1) \nu(d\mathbf{s}) < \infty, \quad (3.17)$$

where $\#\mathbf{s} := \sum_{i=1}^{\infty} \mathbb{1}_{\{s_i > 0\}}$. Denote

$$\mathcal{S}_1 := \{\mathbf{s} \in \mathcal{S} : s_1 > 0, s_2 = s_3 = \dots = 0\},$$

then (3.17) forces that $\nu(\mathcal{S} \setminus \mathcal{S}_1) < \infty$. So the branching rate is finite and on average a finite number of child particles are generated in each splitting event. Denote the number of particles at time $t \geq 0$ by

$$N(t) := \sum_{i=1}^{\infty} \mathbb{1}_{\{X_i(t) \neq 0\}}.$$

Under condition (3.17), the process $(N(t), t \geq 0)$ is simply a *branching process*; see e.g. [2] for basic properties.

We further suppose that

$$\kappa(0) > 0, \quad (3.18)$$

which is known as the *supercritical* condition for the branching process N . It is known (Theorem III.4.1 in [2]) that (3.18) is a sufficient and necessary condition such that the *non-extinction* event

$$\{N(t) > 0 \text{ for all } t \geq 0\}$$

has strictly positive probability.

We next replace (3.17) by a stronger condition

$$\text{there exists } \gamma \in (1, 2], \text{ such that } \int_{\mathcal{S} \setminus \mathcal{S}_1} (\#\mathbf{s})^\gamma \nu(d\mathbf{s}) < \infty. \quad (3.19)$$

This assumption concerns the non-negative martingale (obtained by Proposition 2.13):

$$M_t := e^{-\kappa(0)t} N(t), \quad t \geq 0.$$

Let us recall a well-known martingale convergence result.

Lemma 3.12 (Theorem 5 in [17]). *Suppose that (3.18) and (3.19) hold. Then the martingale M_t converges to a limit M_∞ almost surely and in $L^\gamma(\mathbb{P})$. Further, conditionally on non-extinction, the limit M_∞ is strictly positive.*

In particular, Lemma 3.12 entails that $(M_t)_{t \geq 0}$ is bounded in $L^\gamma(\mathbb{P})$, i.e. there exists $C_\gamma > 0$ such that

$$\sup_{t \geq 0} \mathbb{E}[M_t^\gamma] < C_\gamma. \quad (3.20)$$

Note that (3.19) is also the necessary condition for M_t to have finite γ -moment (Corollary III 6.1 in [2]).

The last assumption is that

$$\int_{\mathcal{S}} \sum_{i=1}^{\infty} \mathbb{1}_{\{0 < s_i < \frac{1}{2}\}} \log(|\log s_i|) d\nu(d\mathbf{s}) < \infty. \quad (3.21)$$

To understand this condition, we state the following statement, which extends Lemma 3.1 in [15] (for the case when ν is binary and conservative).

Lemma 3.13. *For every $\alpha \in \text{dom}(\kappa)$, there exists a Lévy process ξ_α with Laplace exponent*

$$\Phi_\alpha(q) := \kappa(q + \alpha) - \kappa(\alpha), \quad q \geq 0.$$

Specifically, the Lévy process ξ_α has characteristics $(\sigma_\alpha, c_\alpha, \Lambda_\alpha, 0)$, where $\sigma_\alpha := \sigma$,

$$c_\alpha := c + \sigma^2 \alpha + \int_{\mathcal{S}} \left((1 - s_1) - \sum_{i=1}^{\infty} s_i^\alpha (1 - s_i) \right) \nu(d\mathbf{s}),$$

and the Lévy measure Λ_α on $(-\infty, 0)$ is defined such that for every bounded measurable function g on $(-\infty, 0)$ there is

$$\int_{(-\infty, 0)} g(z) \Lambda_\alpha(dz) = \int_{\mathcal{S}} \sum_{i=1}^{\infty} \mathbb{1}_{\{s_i > 0\}} s_i^\alpha g(\log s_i) \nu(d\mathbf{s}).$$

Proof. We first claim that Λ_α is a Lévy measure that satisfies (2.10). Indeed, since $\alpha \in \text{dom}(\kappa)$ and ν satisfies (2.10), we have that

$$\int_{(-\infty, 0)} (1 - e^z)^2 \Lambda_\alpha(dz) = \int_S \sum_{i=1}^{\infty} s_i^\alpha (1 - s_i)^2 \nu(ds) \leq \int_S \sum_{i=2}^{\infty} s_i^\alpha \nu(ds) + \int_S (1 - s_1)^2 \nu(ds) < \infty.$$

We next check that c_α is finite. Notice that $(1 - s_1^\alpha) \leq (\alpha \vee 1)(1 - s_1)$, we hence deduce from (2.10) that

$$\int_S (1 - s_1)(1 - s_1^\alpha) \nu(ds) \leq \int_S (\alpha \vee 1)(1 - s_1)^2 \nu(ds) < \infty.$$

As $\alpha \in \text{dom}(\kappa)$ entails that $\int_S \sum_{i=2}^{\infty} s_i^\alpha \nu(ds) < \infty$, we conclude that $c_\alpha < \infty$. Therefore, there exists a Lévy process ξ_α with characteristics $(\sigma_\alpha, c_\alpha, \Lambda_\alpha, 0)$. It is straightforward to check that ξ_α indeed has Laplace exponent Φ_α , which completes the proof. \square

In particular, if $\kappa(0) < \infty$, then it follows from Lemma 3.13 that

$$\Phi_0(q) := \kappa(q) - \kappa(0), \quad q \geq 0 \quad (3.22)$$

is the Laplace exponent of a certain Lévy process. Then we observe from Lemma 2.1 that (3.21) is the sufficient and necessary condition that an OU type process with characteristics (Φ_0, θ) possesses a unique stationary distribution Π_0 . Let $\tilde{\Pi}_0$ be the image of Π_0 by the map $y \mapsto e^y$, so $\tilde{\Pi}_0$ is a probability measure on \mathbb{R}_+ with finite moments

$$\int_{\mathbb{R}_+} x^q \tilde{\Pi}_0(dx) = \exp \left(\int_0^\infty (\kappa(e^{-\theta s} q) - \kappa(0)) ds \right), \quad q \geq 0.$$

We now state the main result of this section.

Theorem 3.14. *Suppose that (3.18), (3.19) and (3.21) hold. Then for every bounded and continuous function f on \mathbb{R}_+ ,*

$$\lim_{t \rightarrow \infty} e^{-\kappa(0)t} \sum_{i=1}^{\infty} \mathbb{1}_{\{X_i(t) > 0\}} f(X_i(t)) = \langle \tilde{\Pi}_0, f \rangle M_\infty \quad \text{in } L^\gamma(\mathbb{P}). \quad (3.23)$$

Remark 3.15. *It is known (Theorem III.7.2 in [2]) that the martingale M_t converges to M_∞ in $L^1(\mathbb{P})$ if and only if*

$$\int_{S \setminus \mathcal{S}_1} \#s \mathbb{1}_{\{\#s > 0\}} \log(\#s) \nu(ds) < \infty. \quad (3.24)$$

However, when (3.19) is replaced by the weaker condition (3.24), our proof of Theorem 3.14 cannot be extended to prove that the convergence as in (3.23) holds for $L^1(\mathbb{P})$.

As a consequence of Theorem 3.14, we obtain a law of large numbers.

Corollary 3.16 (Law of large numbers). *Suppose that (3.18), (3.19) and (3.21) hold. Then for every bounded and continuous function f on \mathbb{R}_+ , conditionally on non-extinction, there is*

$$\lim_{t \rightarrow \infty} N(t)^{-1} \sum_{i=1}^{\infty} \mathbb{1}_{\{X_i(t) > 0\}} f(X_i(t)) = \langle \tilde{\Pi}_0, f \rangle \quad \text{in probability.}$$

Proof of Corollary 3.16. Conditionally on non-extinction, M_∞ is strictly positive. So it follows from Lemma 3.12 that

$$\lim_{t \rightarrow \infty} \frac{e^{\kappa(0)t}}{N(t)} = M_\infty^{-1} \quad \text{a.s.}$$

Combining this and Theorem 3.14, we deduce the claim. \square

Theorem 3.14 and Corollary 3.16 should be compared with the law of large numbers in branching diffusions [22] and the convergence results of Crump-Mode-Jagers branching processes [31, 23].

Another worthynoting consequence of Theorem 3.14 is about the long-time asymptotic for the solutions of growth-fragmentation equations; see [29] and references therein for similar estimates.

Corollary 3.17. *Suppose that (3.18), (3.19) and (3.21) hold. Let $(\rho_{\mathbf{X}}(t), t \geq 0)$ be a solution to the growth-fragmentation equation (3.1) given by Proposition 3.1, then the probability measure $e^{-\kappa(0)t} \rho_{\mathbf{X}}(t)$ converges weakly to $\tilde{\Pi}_0$. Further, $\tilde{\Pi}_0$ is a solution to the stationary equation: for every $f \in \mathcal{C}_c^\infty(\mathbb{R}_+)$,*

$$\langle \tilde{\Pi}_0, \mathcal{L}f \rangle = \kappa(0)f, \quad (3.25)$$

where \mathcal{L} is as in (3.2).

Proof of Corollary 3.17. Taking expectation to (3.23), we deduce that $e^{-\kappa(0)t} \rho_{\mathbf{X}}(t)$ converges vaguely to $\tilde{\Pi}_0$. We also know that $\rho_{\mathbf{X}}(t)(\mathbb{R}_+) = \mathbb{E}[N(t)] = e^{-\kappa(0)t}$, so $e^{-\kappa(0)t} \rho_{\mathbf{X}}(t)$ is indeed a probability measure and thus the convergence also holds weakly.

It remains to prove that $\tilde{\Pi}_0$ is a solution to (3.25). Since $(\rho_{\mathbf{X}}(t), t \geq 0)$ is a solution to (3.1), we easily check that

$$\frac{\partial}{\partial t} \langle e^{-\kappa(0)t} \rho_{\mathbf{X}}(t), f \rangle = -\kappa(0) \langle e^{-\kappa(0)t} \rho_{\mathbf{X}}(t), f \rangle + \langle e^{-\kappa(0)t} \rho_{\mathbf{X}}(t), \mathcal{L}f \rangle.$$

Letting $t \rightarrow \infty$, we conclude the claim. \square

Open question 3.18. *A natural question is whether the convergence in Theorem 3.14 also holds almost surely. We expect that methods developed in the proof of Theorem 6 in [22] might be of use, that is, by first proving along lattice times, then replacing lattice times with continuous time.*

Open question 3.19. *When (3.17) does not hold, there are almost surely infinitely many fragments at any time, so a law of large numbers as Corollary 3.16 does not make sense. However, it might be interesting to generalize Theorem 3.14 to this case.*

The core of the proof of Theorem 3.14 is the following many-to-one formula.

Lemma 3.20 (Many-to-one formula). *Suppose that (3.17) holds. Let χ be the exponential of an OU type process with characteristics (Φ_0, θ) , where Φ_0 is as in (3.22). For every $t \geq 0$, there is the identity in law*

$$\chi(t) \stackrel{d}{=} e^{-\kappa(0)t} \rho_{\mathbf{X}}(t).$$

Proof. We deduce from Theorem 2.8 that $\mathbb{E}[\chi^q(t)] = \int_{\mathbb{R}_+} x^q e^{-\kappa(0)t} \rho_{\mathbf{X}}(t)(dx)$ for all $q \geq 0$. As the Laplace transform characterizes the law of a random variable, we deduce the identity in law. \square

We are now ready to prove Theorem 3.14.

Proof of Theorem 3.14. Equivalently, we shall prove that for every bounded and continuous function g on \mathbb{R} , we have the convergence

$$\lim_{t \rightarrow \infty} e^{-\kappa(0)t} \sum_{i=1}^{\infty} \mathbb{1}_{\{X_i(t) > 0\}} g(\log X_i(t)) = \langle \Pi_0, g \rangle M_\infty \quad \text{in } L^\gamma(\mathbb{P}).$$

For simplicity, denote

$$U_t := e^{-\kappa(0)t} \sum_{i=1}^{\infty} \mathbb{1}_{\{X_i(t) > 0\}} g(\log X_i(t)), \quad t \geq 0.$$

Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration of \mathbf{X} , then it suffices to prove that

$$\lim_{t \rightarrow \infty} \sup_{s \geq 0} |U_{t+s} - \mathbb{E}[U_{t+s} \mid \mathcal{F}_t]| = 0 \quad \text{in } L^\gamma(\mathbb{P}), \quad (3.26)$$

and that there exists a function $t \mapsto S(t) > 0$ such that

$$\lim_{t \rightarrow \infty} \mathbb{E}[U_{t+S(t)} \mid \mathcal{F}_t] = \langle \Pi_0, g \rangle M_\infty \quad \text{in } L^\gamma(\mathbb{P}). \quad (3.27)$$

We start with (3.26). Let $(\mathbf{X}^{(i)} := (X_1^{(i)}(t), X_2^{(i)}(t), \dots)_{t \geq 0}, i \geq 1)$ be i.i.d. copies of \mathbf{X} , then using the Markov property (Proposition 2.12), we have for every $s \geq 0$ the identity in law:

$$U_{t+s} - \mathbb{E}[U_{t+s} \mid \mathcal{F}_t] \stackrel{d}{=} e^{-\kappa(0)(t+s)} \sum_{i=1}^{\infty} \mathbb{1}_{\{X_i(t) > 0\}} (Y_i(t, s) - \mathbb{E}[Y_i(t, s) \mid \mathcal{F}_t]), \quad (3.28)$$

where

$$Y_i(t, s) := \sum_{j=1}^{\infty} \mathbb{1}_{\{X_j^{(i)}(s) > 0\}} g(e^{-\theta s} \log X_i(t) + \log X_j^{(i)}(s)).$$

Let us now recall a useful inequality (Lemma 1 in [17]): if $\gamma \in [1, 2]$ and (Z_i) are independent random variables with each $\mathbb{E}[Z_i] = 0$, then for every $n \in \mathbb{N} \cup \{\infty\}$ there is

$$\mathbb{E} \left[\left| \sum_{i=1}^n Z_i \right|^\gamma \right] \leq 2^\gamma \sum_{i=1}^n \mathbb{E} \left[|Z_i|^\gamma \right]. \quad (3.29)$$

Since $Z_i := Y_i(t, s) - \mathbb{E}[Y_i(t, s) \mid \mathcal{F}_t]$ are independent conditionally on \mathcal{F}_t , applying (3.29) to (3.28), we have

$$\mathbb{E} \left[\left| U_{t+s} - \mathbb{E}[U_{t+s} \mid \mathcal{F}_t] \right|^\gamma \right] \leq 2^\gamma e^{-\gamma \kappa(0)(t+s)} \sum_{i=1}^{\infty} \mathbb{E} \left[\mathbb{1}_{\{X_i(t) > 0\}} \left| Y_i(t, s) - \mathbb{E}[Y_i(t, s) \mid \mathcal{F}_t] \right|^\gamma \right].$$

For every $i \in \mathbb{N}$, using Jensen's inequality (the finite form) and then conditional Jensen's inequality, we find that

$$\begin{aligned} & \mathbb{E} \left[\mathbb{1}_{\{X_i(t) > 0\}} \left| Y_i(t, s) - \mathbb{E}[Y_i(t, s) \mid \mathcal{F}_t] \right|^\gamma \right] \\ & \leq 2^{\gamma-1} \mathbb{E} \left[\mathbb{1}_{\{X_i(t) > 0\}} (|Y_i(t, s)|^\gamma + |\mathbb{E}[Y_i(t, s) \mid \mathcal{F}_t]|^\gamma) \right] \leq 2^\gamma \mathbb{E} \left[\mathbb{1}_{\{X_i(t) > 0\}} |Y_i(t, s)|^\gamma \right]. \end{aligned}$$

By conditioning on \mathcal{F}_t and using (3.20), we deduce that

$$\mathbb{E} \left[\mathbb{1}_{\{X_i(t) > 0\}} |Y_i(t, s)|^\gamma \right] \leq \|g\|_\infty^\gamma \mathbb{E} \left[\mathbb{1}_{\{X_i(t) > 0\}} \left(\sum_{j=1}^{\infty} \mathbb{1}_{\{X_j^{(i)}(s) > 0\}} \right)^\gamma \right] \leq \|g\|_\infty^\gamma C_\gamma e^{\gamma \kappa(0)s} \mathbb{E} \left[\mathbb{1}_{\{X_i(t) > 0\}} \right].$$

Summarizing, for every $s, t > 0$ we have

$$\mathbb{E} [|U_{t+s} - \mathbb{E}[U_{t+s} \mid \mathcal{F}_t]|^\gamma] \leq 2^{2\gamma} \|g\|_\infty^\gamma C_\gamma e^{-(\gamma-1)\kappa(0)t},$$

which converges to 0 as $t \rightarrow \infty$, since $\gamma > 1$. So we have justified (3.27).

It remains to prove (3.27). Recall that

$$\mathbb{E}[U_{t+s} \mid \mathcal{F}_t] = e^{-\kappa(0)(t+s)} \sum_{i=1}^{\infty} \mathbb{1}_{\{X_i(t) > 0\}} \mathbb{E}[Y_i(t, s) \mid \mathcal{F}_t].$$

Applying the many-to-one formula (Lemma (3.20)) to $\mathbb{E}[Y_i(t, s) \mid \mathcal{F}_t]$ yields

$$e^{-\kappa(0)s} \mathbb{E}[Y_i(t, s) \mid \mathcal{F}_t] = \mathbb{E} \left[g(e^{-\theta s} \log x_i + \log \chi(s)) \right] \Big|_{x_i = X_i(t)}, \quad (3.30)$$

where χ is the exponential of an OU type process with characteristics (Φ_0, θ) . Consider a family of increasing compact sets $K_t \uparrow (0, \infty)$, say $K_t := [t^{-1}, t]$. On the one hand, if we only consider those $X_i(t) \notin K_t$, then it follows from (3.30) that

$$\sup_{s \geq 0} \left| e^{-\kappa(0)(t+s)} \sum_{i=1}^{\infty} \mathbb{1}_{\{X_i(t) \notin K_t\}} \mathbb{E}[Y_i(t, s) \mid \mathcal{F}_t] \right| \leq \|g\|_{\infty} e^{-\kappa(0)t} \sum_{i=1}^{\infty} \mathbb{1}_{\{X_i(t) \notin K_t\}}.$$

By the many-to-one formula (Lemma 3.20), the right-hand-side has mean value

$$\|g\|_{\infty} \mathbb{P}(\chi(t) \notin K_t),$$

which converges to zero as $t \rightarrow \infty$. As (3.20) holds, we have by the dominated convergence that

$$\lim_{t \rightarrow \infty} \sup_{s \geq 0} \left| e^{-\kappa(0)(t+s)} \sum_{i=1}^{\infty} \mathbb{1}_{\{X_i(t) \notin K_t\}} \mathbb{E}[Y_i(t, s) \mid \mathcal{F}_t] \right| = 0 \quad \text{in } L^{\gamma}(\mathbb{P}). \quad (3.31)$$

On the other hand, since g is uniformly continuous on any compact set K on \mathbb{R} and $\theta > 0$, we deduce by Lemma 2.1 that the following convergence holds uniformly for $x \in K$:

$$\lim_{s \rightarrow \infty} \mathbb{E} \left[g(e^{-\theta s} x + \log \chi(s)) \right] = \langle \Pi_0, g \rangle.$$

Then using (3.30) and Lemma 3.12, we can choose $S(t) > 0$, depending on K_t , such that

$$\lim_{t \rightarrow \infty} e^{-\kappa(0)(t+S(t))} \sum_{i=1}^{\infty} \mathbb{1}_{\{X_i(t) \in K_t\}} \mathbb{E}[Y_i(t, S(t)) \mid \mathcal{F}_t] = \langle \Pi_0, g \rangle M_{\infty} \quad \text{in } L^{\gamma}(\mathbb{P}). \quad (3.32)$$

Combining (3.31), (3.32), we then deduce (3.27), which completes the proof. \square

4 Connections with Markovian growth-fragmentation processes

In this section, we first present *Markovian growth-fragmentation processes* [10] associated with exponential OU type processes, and then study their connections with OU type growth-fragmentations.

4.1 Markovian growth-fragmentations associated with exponential OU type processes

Throughout this section, let ξ be a spectrally negative Lévy process with characteristics (σ, c, Λ, k) , Z be an OU type process with index θ driven by ξ as in (2.3), and

$$X(t) := \exp(Z(t)), \quad t \geq 0.$$

For every $x > 0$, write P_x for the law of X starting from $X(0) = x$. Recall that the Laplace exponent Φ of ξ is given by (2.1). We introduce $\kappa: [0, \infty) \rightarrow (-\infty, \infty]$ by

$$\kappa(q) := \Phi(q) + \int_{(-\infty, 0)} (1 - e^y)^q \Lambda(dy), \quad q \geq 0.$$

Then $\kappa \geq \Phi$, κ is convex and $\kappa(q) < \infty$ for all $q \geq 2$ because of (2.2). The function κ shall be referred to as the **cumulant** of ξ or Z or X ; we shall later see that κ indeed plays a similar role as the cumulant of an OU type growth-fragmentation defined as in (2.11). We stress that κ does not characterize the law of ξ , see Lemma 2.1 in [34]. The cumulant κ also plays a crucial role in the study of self-similar growth-fragmentations, see [10, 34].

For future use, we state the following property of X . Define a function $F : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ by

$$F(t, x) := x^{2\exp(\theta t)} F_1(t) F_\eta(t), \quad t \geq 0, x > 0, \quad (4.1)$$

where

$$F_1(t) := \exp\left(-\int_0^t \Phi(2e^{\theta r}) dr\right)$$

and

$$F_\eta(t) := \exp\left(-\int_0^t \eta^{-1} \left(\kappa(2e^{\theta r}) - \Phi(2e^{\theta r})\right) dr\right),$$

with a constant $\eta \in (0, 1)$. Note that F_η is non-increasing.

Lemma 4.1. *For every $x > 0$ and $s, t \geq 0$, we have*

$$\mathbb{E}_x \left[F(s+t, X(t)) + \sum_{0 \leq r \leq t} F(s+r, -\Delta X(r)) \right] \leq F(s, x). \quad (4.2)$$

and

$$\mathbb{E}_x \left[\sum_{0 \leq r} F(s+r, -\Delta X(r)) \right] \leq \eta F(s, x). \quad (4.3)$$

Proof. Applying (2.4) with $q = 2\exp(\theta(t+s))$, we have for every $s \geq 0$ that

$$\mathbb{E}_x \left[F(s+t, X(t)) \right] = x^{2\exp(\theta s)} \exp\left(\int_0^t \Phi(2e^{\theta(t+s-r)}) dr\right) F_1(t) F_\eta(t) = x^{2\exp(\theta s)} F_1(s) F_\eta(s+t) \quad (4.4)$$

As (2.3) shows that

$$-\Delta X(r) = X(r-)(1 - e^{\Delta \xi(r)}),$$

applying the compensation formula (see e.g. [7]) to the Poisson point process $\Delta \xi$, we have that

$$\begin{aligned} & \mathbb{E}_x \left[\sum_{0 \leq r \leq t} F(s+r, -\Delta X(r)) \right] \\ &= \int_0^t \mathbb{E}_x [F(s+r, X(r))] dr \int_{(-\infty, 0)} (1 - e^z)^{2\exp(\theta(s+r))} \Lambda(dz) \\ &= \int_0^t x^{2\exp(\theta s)} F_1(s) F_\eta(s+r) \left(\kappa(2e^{\theta(s+r)}) - \Phi(2e^{\theta(s+r)}) \right) dr \\ &= \eta x^{2\exp(\theta s)} F_1(s) (F_\eta(s) - F_\eta(s+t)) \end{aligned} \quad (4.5)$$

where we have used (4.4) in the second equality. Adding (4.4) to (4.5) and using the fact that F_η is non-increasing, we obtain (4.2). Letting $t \rightarrow \infty$ in (4.5), we also have (4.3). \square

Lemma 4.1 enables us to list the jump times of X as a sequence $(t_i, i \in \mathbb{N})$ such that $(F(|\Delta X(t_i)|, t_i), i \in \mathbb{N})$ is decreasing. In the sequel, **the i -th jump time of X** shall always refer to the i -th element t_i in this sequence.

A Markovian growth-fragmentation process associated with X can be constructed by using the approach in [10, 34]. We first construct a **cell system driven by X** , which is a family of processes indexed by the

Ulam-Harris tree $\mathbb{U} := \bigcup_{i=0}^{\infty} \mathbb{N}^i$,

$$\mathcal{X} := (\mathcal{X}_u, u \in \mathbb{U}),$$

where each \mathcal{X}_u depicts the evolution of the size of the cell indexed by u as time passes. Specifically, The ancestor cell \emptyset is born at $b_\emptyset := 0$ with initial size 1, and the life career $\mathcal{X}_\emptyset = (\mathcal{X}_\emptyset(t), t \geq 0)$ is an OU type process of law P_1 . The laws of the first generation $\mathbb{N} \subset \mathbb{U}$ are determined by the trajectory of \mathcal{X}_\emptyset : for $i \in \mathbb{N}$, say the i -th jump time of \mathcal{X}_\emptyset occurs at time t_i and has size $x_i := -\Delta \mathcal{X}_\emptyset(t_i)$, we then set $b_i = t_i$ and build a sequence of conditional independent processes $(\mathcal{X}_i)_{i \in \mathbb{N}}$ with respective conditional distribution P_{x_i} . We continue in this way to construct higher generations recursively: For every individual $u \in \mathbb{U}$, the laws of her daughters are determined by the trajectory of \mathcal{X}_u : given \mathcal{X}_u , say the i -th jump of \mathcal{X}_u is at time t with $y := -\Delta \mathcal{X}_u(t)$, then its i -th daughter ui is born at time $b_{ui} := t$ and ui 's size process $\mathcal{X}_{ui} = (\mathcal{X}_{ui}(r), r \geq 0)$ has conditional distribution P_y , independent of the size processes of the other individuals in the same generation. The above description uniquely determine the law of the cell system \mathcal{X} , denoted by \mathcal{P} .

Lemma 4.2. *Let F be a function as in (4.1). For every $t \geq 0$,*

$$\mathcal{P} \left[\sum_{u \in \mathbb{U}, b_u \leq t} F(t, \mathcal{X}_u(t - b_u)) \right] \leq 1.$$

Proof. As (4.2) holds, the claim follows from Lemma 3.2 in [34]. \square

In particular, this lemma implies that at every time $t \geq 0$, we can rank the sizes of the cells alive at t , i.e.

$$\{\!\!\{ \mathcal{X}_u(t - b_u) : u \in \mathbb{U}, b_u \leq t \}\!\!\},$$

in decreasing order and obtain a sequence in $\ell^{2e^{\theta t} \downarrow}$ denoted by $\mathbf{X}(t)$. We refer to $\mathbf{X} = (\mathbf{X}(t), t \geq 0)$ as a **(Markovian) growth-fragmentation process driven by X** . Write \mathbf{P} for the law of \mathbf{X} under \mathcal{P} .

By construction, the law of \mathbf{X} is determined by the law of X . However, growth-fragmentations driven by cell processes with different laws may have the same distribution. In [34] a necessary and sufficient condition that characterizes the law of the growth-fragmentation when the cell process is a self-similar Markov process is given. By adapting the approach in [34], we find a family of OU type processes which give rise to the same (in law) growth-fragmentation.

Lemma 4.3. *Let \tilde{Z} be an OU type process with characteristics $(\tilde{\Phi}, \theta)$ and $\tilde{X} := \exp(\tilde{Z})$. Suppose that \tilde{X} has the same cumulant κ , then the growth-fragmentations \mathbf{X} and $\tilde{\mathbf{X}}$, driven respectively by X and \tilde{X} , have the same finite-dimensional distributions.*

Proof. In order to apply Theorem 3.7 in [34], we introduce the following manipulation. Since X and \tilde{X} have the same cumulant κ , by Proposition 2.5 in [34] we can build a pair of spectrally negative Lévy processes ξ and $\tilde{\xi}$ with respective Laplace exponents Φ and $\tilde{\Phi}$, such that ξ is a *switching transform* of $\tilde{\xi}$, see Lemma 2.2 in [34] for the precise meaning. In particular, we have that the switching time $\tau := \inf \left\{ t \geq 0 : \xi(t) \neq \tilde{\xi}(t) \right\}$ is almost surely strictly positive and

$$\exp(\xi(\tau)) + \exp(\tilde{\xi}(\tau)) = \exp(\xi(\tau-)).$$

We may assume $\log X$ and $\log \tilde{X}$ (both starting from 0) are OU type processes associated respectively with ξ and $\tilde{\xi}$ by (2.3), then $\inf \left\{ t \geq 0 : X(t) \neq \tilde{X}(t) \right\}$ is equal to τ and $X(\tau) + \tilde{X}(\tau) = X(\tau-) = \tilde{X}(\tau)$. Let \tilde{X}' be an independent copy of \tilde{X} and set

$$\tilde{X}''(t) := \tilde{X}(t) \mathbb{1}_{\{t < \tau\}} + \tilde{X}(\tau)^{\exp(-\theta(t-\tau))} \tilde{X}'(t - \tau) \mathbb{1}_{\{t \geq \tau\}}, \quad t \geq 0.$$

Using (2.3) and the strong Markov property of an OU type process, one easily checks that $\tilde{X}'' \stackrel{d}{=} \tilde{X}$ and further the couple (X, \tilde{X}'') satisfies the following properties:

(B1) Let $\tau := \inf\{t \geq 0 : X(t) \neq \tilde{X}(t)\}$. There is almost surely either $\tau = \infty$ or the identity

$$X(\tau) + \tilde{X}''(\tau) = X(\tau-) = \tilde{X}''(\tau-).$$

(B2) (Asymmetric Markov branching property) Conditionally given $\tau > t$, the process

$$(X(r+t)X(t)^{-\exp(-\theta t)}, \tilde{X}''(r+t)\tilde{X}''(t)^{-\exp(-\theta t)})_{r \geq 0}$$

is a copy of (X, \tilde{X}) ; conditionally given $\tau \leq t$, the two processes $(X(r+t)X(t)^{-\exp(-\theta t)})_{r \geq 0}$ and $(\tilde{X}''(r+t)\tilde{X}''(t)^{-\exp(-\theta t)})_{r \geq 0}$ are independent, and have the laws of X and \tilde{X}'' respectively.

Therefore, we find that (X, \tilde{X}'') is a *bifurcator* in the sense of Definition 3.7 in [34]. Combining this and Lemma 4.1, we check that the conditions of Theorem 3.7 in [34] are fulfilled, then it follows that \mathbf{X} and $\tilde{\mathbf{X}}$ have the same finite-dimensional distributions. \square

4.2 Binary OU type growth-fragmentations

Definition 4.4. A OU type growth-fragmentation process is **binary**, if its dislocation measure ν has support on

$$\{\mathbf{s} \in \mathcal{S} : s_1 + s_2 = 1, s_3 = s_4 = \dots = 0\} \cup \{(0, 0, \dots)\}. \quad (4.6)$$

In this subsection we study the relation between Markovian growth-fragmentations and binary OU type growth-fragmentation processes. We first observe that each binary OU type growth-fragmentation can be viewed as a Markovian growth-fragmentation in the following sense.

Lemma 4.5. Let \mathbf{X} be a binary OU type growth-fragmentation and X_* be the selected fragment of \mathbf{X} . Then \mathbf{X} is a Markovian growth-fragmentation associated with X_* .

Proof. This proof is an adaptation of arguments in the proof of Proposition 3 in [10]. Recall from Lemma 2.11 that the select fragment of \mathbf{X} is obtained by keeping the larger child and discarding the smaller one at each dislocation, and its size X_* evolves as the exponential of an OU type process. Each jump time $t \geq 0$ of X_* corresponds to a dislocation, in which a fragment of size $X_*(t-)$ splits into two children, with the larger one of size $X_*(t)$ and the smaller one of size $-\Delta X_*(t)$. Further, consider for every $\ell > 0$ the truncated system $\mathbf{X}^{(\ell)}$ (see Lemma 2.5 and Definition 2.7). We find by (2.9) that the smaller fragment is kept in $\mathbf{X}^{(\ell)}$, if and only if $\frac{|\Delta X_*(t)|}{X_*(t-)} > e^{-\ell}$.

Therefore, the dynamics of $\mathbf{X}^{(\ell)}$ can be described in the following way. Let P_x be the law of the process $(x^{\exp(-\theta t)} X_*(t))_{t \geq 0}$. The ancestor of the cell system is the selected fragment X_* of law P_1 . At each time $t \geq 0$ when $\frac{|\Delta X_*(t)|}{X_*(t-)} > e^{-\ell}$, a child cell is born with initial size $y := -\Delta X_*(t)$. The size of the child particle proceeds according the selected fragment of the sub-population, so it has the law of P_y . These children form the first generation, which evolve independently one of the others. Iterating this argument, we produce all generations and conclude that $\mathbf{X}^{(\ell)}$ has the same law as a cell system associated with X_* , in which each child cell (together with its descendants) is killed whenever its size at birth is less than or equal to $e^{-\ell}$ times the size of the parent right before the birth of child. Letting $\ell \rightarrow \infty$, the claim follows from the monotonicity. \square

Corollary 4.6. The law of a binary OU type growth-fragmentation \mathbf{X} is characterized by (κ, θ) .

Proof. Suppose that another binary OU type growth-fragmentation $\tilde{\mathbf{X}}$ also has index θ and cumulant κ . Using the binary condition (4.6) and Lemma 2.11, we deduce that the respective selected fragments of $\tilde{\mathbf{X}}$ and \mathbf{X} have the same law. Then it follows from Lemma 4.5 that $\tilde{\mathbf{X}}$ and \mathbf{X} are the same (in law) OU type growth-fragmentation. Conversely, if an OU type growth-fragmentation $\tilde{\mathbf{X}}$ have the same law as \mathbf{X} , then it follows directly from (2.14) and the scaling property (P2) that $\tilde{\mathbf{X}}$ and \mathbf{X} have the same index θ and cumulant κ . \square

Conversely, each Markovian growth-fragmentation driven by an exponential OU type process is a binary OU type growth-fragmentation.

Proposition 4.7. *Let Z be an OU type process with index θ and cumulant κ . Then the Markovian growth-fragmentation $\mathbf{X} := (X_1(t), X_2(t), \dots)_{t \geq 0}$ associated with $\exp(Z)$ is a version of an binary OU type growth-fragmentation characterized by (κ, θ) . In particular, \mathbf{X} possesses a càdlàg version in c_o^\downarrow and for every $t \geq 0$ and $q \geq 2(1 \vee e^{\theta t})$*

$$\mathbb{E} \left[\sum_{i=1}^{\infty} X_i(t)^q \right] = \exp \left(\int_0^t \kappa(qe^{-\theta s}) ds \right) < \infty.$$

Proof. Consider the selected fragment X_* of a binary OU type growth-fragmentation characterized by (κ, θ) . It follows from Lemma 2.11 that $\log X_*$ is an OU type process with cumulant κ . Combining Lemma 4.3 and Lemma 4.5, we deduce that \mathbf{X} has the same finite dimensional distributions as a binary OU type growth-fragmentation characterized by (κ, θ) . We complete the proof by applying Theorem 2.8 to \mathbf{X} . \square

Remark 4.8. *Write $(\sigma, c, \Lambda, k, \theta)$ for the characteristics of Z , then the Markovian growth-fragmentation \mathbf{X} associated with $\exp(Z)$ is an OU type growth-fragmentation with characteristics*

$$\left(\sigma, c - k + \int_{(-\infty, -\log 2)} (1 - 2e^y) \Lambda(dy), \nu_2 + k\delta_{(0,0,\dots)}, \theta \right),$$

where ν_2 is the image of Λ by the map $z \mapsto (\max(e^z, 1 - e^z), \min(e^z, 1 - e^z), 0, \dots)$. Indeed, this OU type growth-fragmentation is binary and has the same cumulant as Z .

Remark 4.9. *Let \tilde{X} be an OU type process with characteristics $(\tilde{\Phi}, \tilde{\theta})$, \mathbf{X} and $\tilde{\mathbf{X}}$ be two Markovian growth-fragmentations driven respectively by X and \tilde{X} . Then the following statements are equivalent:*

- (i) $\kappa = \tilde{\kappa}$ and $\theta = \tilde{\theta}$;
- (ii) X and \tilde{X} can be coupled to form a bifurcator that satisfies (B1) and (B2) in the proof of Lemma 4.3;
- (iii) the growth-fragmentations \mathbf{X} and $\tilde{\mathbf{X}}$ have the same finite dimensional distributions.

Indeed, we have already obtained “(i) \Rightarrow (ii)” and “(ii) \Rightarrow (iii)” from the proof of Lemma 4.3. The implication “(iii) \Rightarrow (i)” follows from Proposition 4.7. This is an analogous result of Theorem 1.1 (for homogeneous growth-fragmentations) and Theorem 1.2 (for self-similar growth-fragmentations) in [34].

5 A connection with random recursive trees

In this section we lift from [5] a certain OU type growth-fragmentation that appears in the destruction of an infinite recursive tree. See also [30] for a related work.

An *infinite recursive tree* is a random rooted tree with vertices indexed by \mathbb{N} , constructed recursively in the following way. We start with linking the vertex 1 (the root) to the vertex 2 by an edge denoted by e_2 . Then we proceed by induction. For $i \geq 2$, vertex i attaches to a vertex chosen uniformly from $\{1, \dots, i-1\}$, say j , by an edge e_i .

We destroy the infinite recursive tree by associating each e_i with an independent exponential clock and breaking each edge when its clock rings. Then the vertices of this tree split into different connected clusters. Let $\Pi(t) = (\Pi_1(t), \Pi_2(t), \dots)$ be the resulting partition of \mathbb{N} at time $t \geq 0$, such that each $\Pi_i(t)$ is the set of the vertices of a cluster at time t , and they are listed in increasing order of the smallest element of the cluster. It has been proven in [5] that

$$W_i(t) := \lim_{n \rightarrow \infty} n^{-e^{-t}} \#\{k \leq n : k \in \Pi_i(t)\} \quad \text{exists for every } i \in \mathbb{N}.$$

Further, $(W_i(t), i \in \mathbb{N})$ can be rearranged in decreasing order, which produces a sequence denoted by $\mathbf{X}^R(t)$. Partial results of Proposition 2.3 and Theorem 3.1 in [5] can be rewritten in our terms as follows.

Proposition 5.1 ([5]). *The process \mathbf{X}^R is a binary OU type growth-fragmentation with characteristics $(\kappa_R, 1)$ in the sense of Corollary 4.6, where*

$$\kappa_R(q) = q\psi(q+1) + (q-1)^{-1}, \quad q > 1,$$

with ψ denoting the digamma function, that is the logarithmic derivative of the gamma function. Equivalently, \mathbf{X}^R has characteristics $(0, -\gamma + 2\log 2, \nu, 1)$, where $\gamma = 0.57721\dots$ is the Euler-Mascheroni constant, and the binary dislocation measure ν is specified by

$$\nu(ds_1) = (s_1^{-2} + (1-s_1)^{-2}) ds_1, \quad \frac{1}{2} \leq s_1 < 1.$$

Then by Proposition 2.12 and Theorem 2.8, we recover immediately Theorem 3.4 in [5], which states the Markov property of \mathbf{X}^R and that for every $t \geq 0$ and $q > e^t$, there is

$$\mathbb{E} \left[\sum_{i=1}^{\infty} X_i^R(t)^q \right] = \frac{q-1}{e^{-t}q-1} \frac{\Gamma(q)}{\Gamma(e^{-t}q)}. \quad (5.1)$$

Indeed, by the property of the digamma function ψ , an easy calculation shows that

$$\exp \left(\int_0^t \kappa_R(e^{-s}q) ds \right) = \frac{\Gamma(q+1)}{\Gamma(e^{-t}q+1)} \frac{q-1}{e^{-t}q-1} \frac{e^{-t}q}{q} = \frac{q-1}{e^{-t}q-1} \frac{\Gamma(q)}{\Gamma(e^{-t}q)}.$$

Then (5.1) follows from Theorem 2.8.

For the readers' convenience, let us briefly justify Proposition 5.1 by using results in [5].

Proof of Proposition 5.1. Let ξ be a spectrally negative Lévy process with characteristics $(0, -\gamma + 1, \Lambda, 0)$, where $\gamma = 0.57721\dots$ is the Euler-Mascheroni constant, and the Lévy measure Λ has density

$$\Lambda(dz) = e^z(1-e^z)^{-2}dz, \quad z \in (-\infty, 0).$$

We know from [5] that the Laplace exponent of ξ is $\Phi_R(q) := q\psi(q+1)$.³ We also have that

$$\int_{-\infty}^0 (1-e^z)^q e^z(1-e^z)^{-2}dz = \frac{1}{q-1}, \quad q > 1.$$

So ξ has cumulant κ_R .

Write P_x for the law of an exponential OU type process X with characteristics $(\Phi_R, 1)$ starting from $x > 0$, then we shall prove that \mathbf{X}^R is Markovian growth-fragmentation associated with X . In this direction, let

³The Lévy-Khintchine formula in [5] has a compensation term different from (2.1), so the drift coefficient is changed.

us consider a cell system \mathcal{X} described as follows. Set the Eve process $\mathcal{X}_\emptyset := W_1$, the weight process of the cluster Π_1 (that contains the root 1). Then \mathcal{X}_\emptyset has distribution P_1 by Theorem 3.1 in [5]. At each jump time of \mathcal{X}_\emptyset , say $s > 0$, the partition process Π has a dislocation in which the block $\Pi_1(s)$ splits into B_1 and B_2 , with B_1 being the block that contains 1. Write Π^{B_2} for the partition process constrained to B_2 and let $y := \lim_{n \rightarrow \infty} n^{-e^{-s}} \#\{i \leq n : i \in B_2\}$, then we deduce by Proposition 2.3 and Theorem 3.1 in [5] that the weight process

$$W_1^{B_2}(t) := \lim_{n \rightarrow \infty} n^{-e^{-(t+s)}} \#\{i \leq n : i \in \Pi_1^{B_2}(t+s)\}, \quad t \geq 0$$

has conditional distribution P_y given \mathcal{X}_\emptyset . We thus view $W_1^{B_2}$ as the daughter process born at the jump time s of \mathcal{X}_\emptyset . In this way we associate each jump time of \mathcal{X}_\emptyset with a daughter; these daughters are independent one of the others, and form the first generation of the cell system. By iteration of this argument, we obtain a cell system driven by X and hence deduce that \mathbf{X}^R is a Markovian growth-fragmentation associated with X . So we know from Proposition 4.7 that \mathbf{X}^R is a binary OU type growth-fragmentation process with characteristics $(\kappa_R, 1)$. \square

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